MIMO Mobile Radio Systems

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Topics

1st lesson:

- introduction
- system modelling
- channel capacity

2nd lesson:

- channel models

3rd lesson:

- canonical system implementation
- signal processing with non cooperative inputs (BLAST)
- signal processing with non cooperative outputs
- diversity

1. Introduction

Application of MIMO: 802.11n

characteristics:

- 20 MHz to 40 MHz bandwidth
- OFDM
- 2 to 4 antennas per station
- spatial multiplexing
- up to 540 Mbit/s

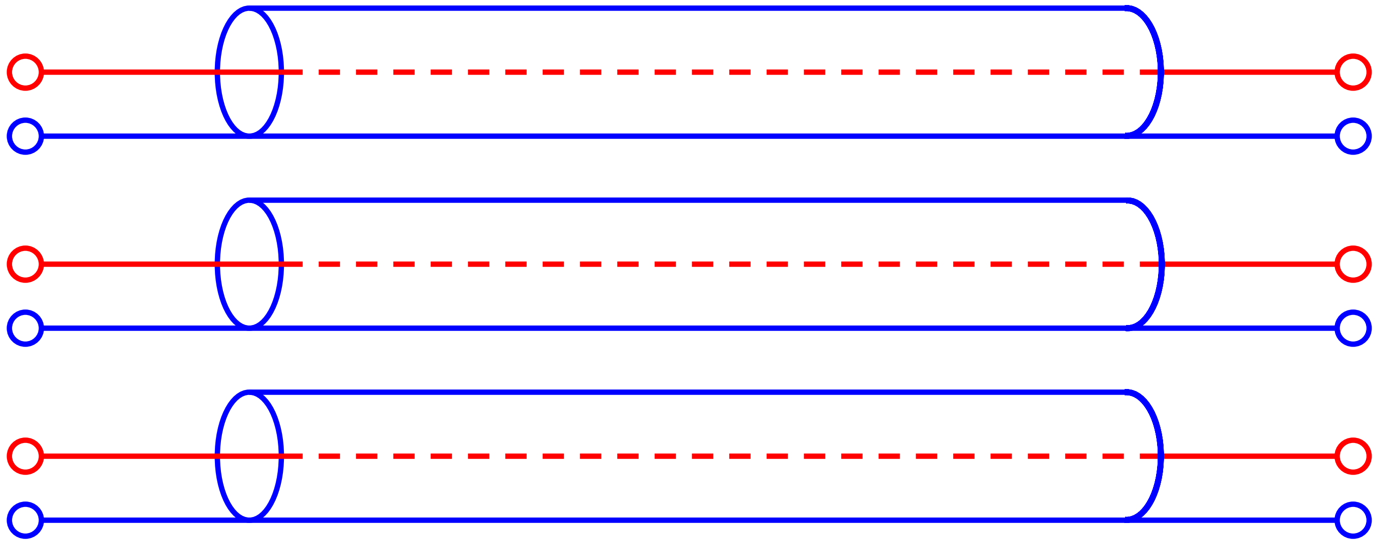
Terms

Czech:
mimo = except

		number of inputs	
		$N = 1$	$N > 1$
number of outputs	$M = 1$	SISO	MISO
	$M > 1$	SIMO	(N, M) MIMO

$\Rightarrow$ spatial signal processing

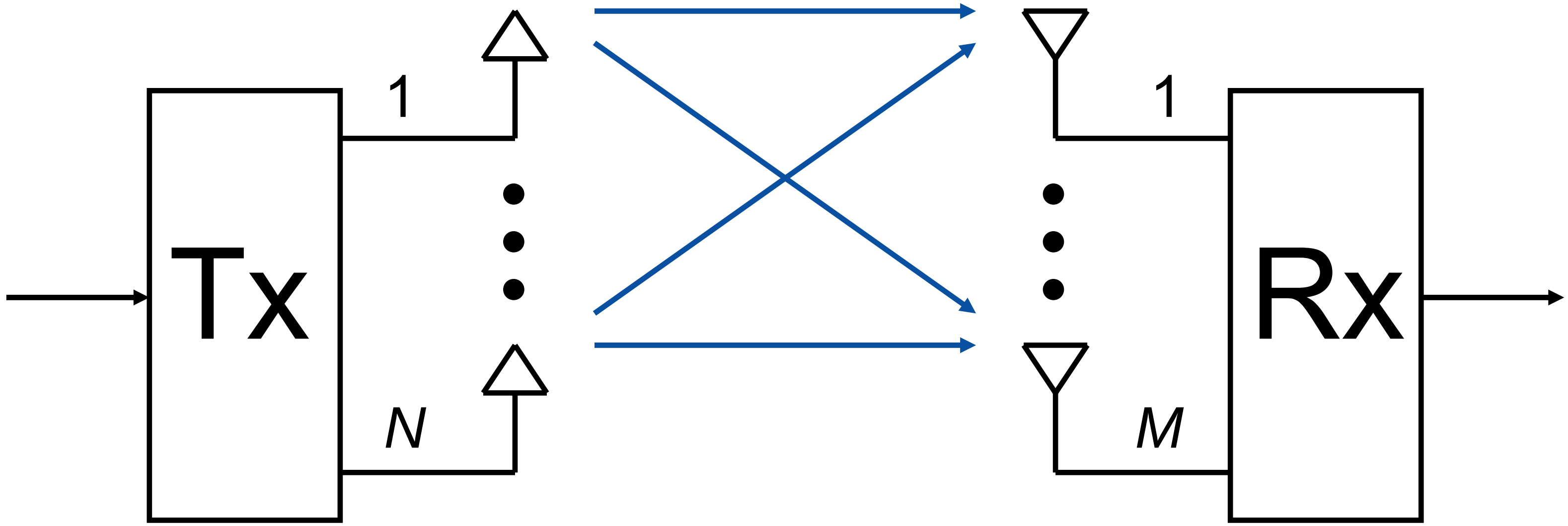
Cable Binder



R parallel wires ⇒ (*R*, *R*) MIMO system
capacity proportional to *R*
(for fixed transmitted power per input)

insufficient shielding ⇒ cross couplings

Multiple Antenna System



increasing the capacity by increasing the bandwidth is expensive (UMTS in Germany: approx. 50 000 000 000 € for 120 MHz)

alternative: use spectrum more efficiently

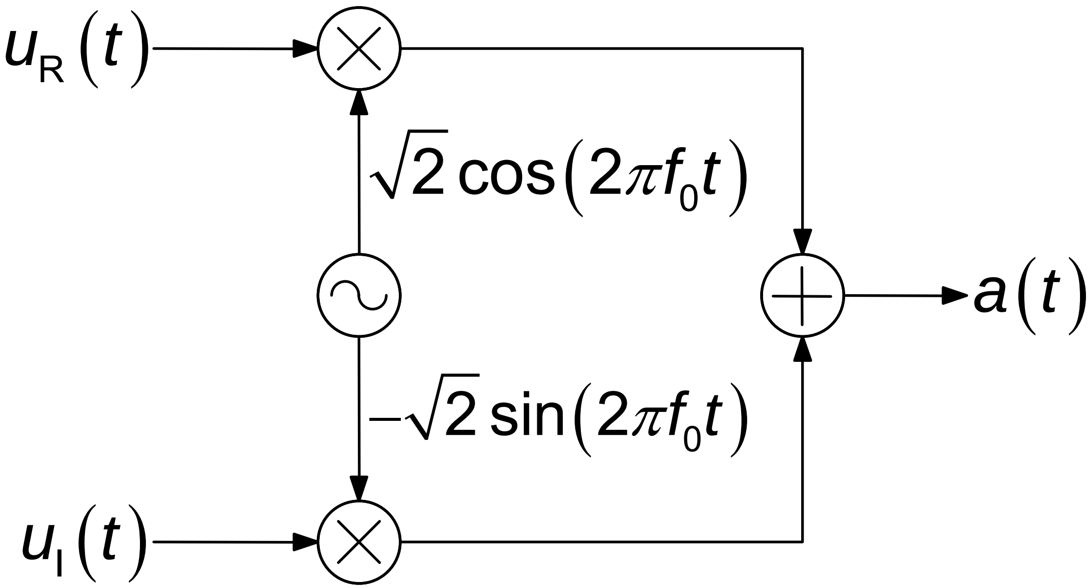
2. System Modelling

Quadrature Modulator

$$a(t) = \sqrt{2}\,\mathrm{Re}\left\{\underline{u}(t)\cdot \mathrm{e}^{\mathrm{j}2\pi f_0 t}\right\}$$

$$= \sqrt{2}u_\mathrm{R}(t)\cdot\cos(2\pi f_0 t) - \sqrt{2}u_\mathrm{I}(t)\cdot\sin(2\pi f_0 t)$$

in-phase component (u_R) quadrature component (u_I)



$a(t)$: bandpass signal

$\underline{u}(t)$: equivalent lowpass signal

Quadrature Demodulator

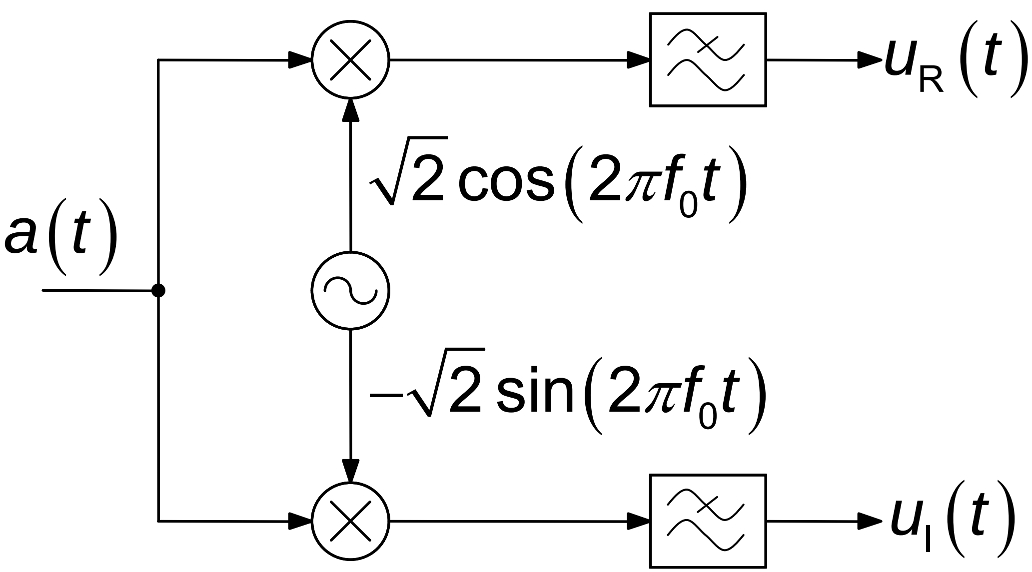
$$a(t)\cdot\sqrt{2}e^{-j2\pi f_0 t} = \left[\sqrt{2}u_R(t)\cdot\cos(2\pi f_0 t) - \sqrt{2}u_I(t)\cdot\sin(2\pi f_0 t)\right]$$

$$\cdot\left[\sqrt{2}\cos(2\pi f_0 t) - j\sqrt{2}\sin(2\pi f_0 t)\right]$$

$$= u_R(t) + u_R(t)\cos(4\pi f_0 t) - u_I(t)\sin(4\pi f_0 t)$$

$$+ju_I(t) - ju_I(t)\cos(4\pi f_0 t) - ju_R(t)\sin(4\pi f_0 t)$$

high frequency



WLAN Chip

Time Shift

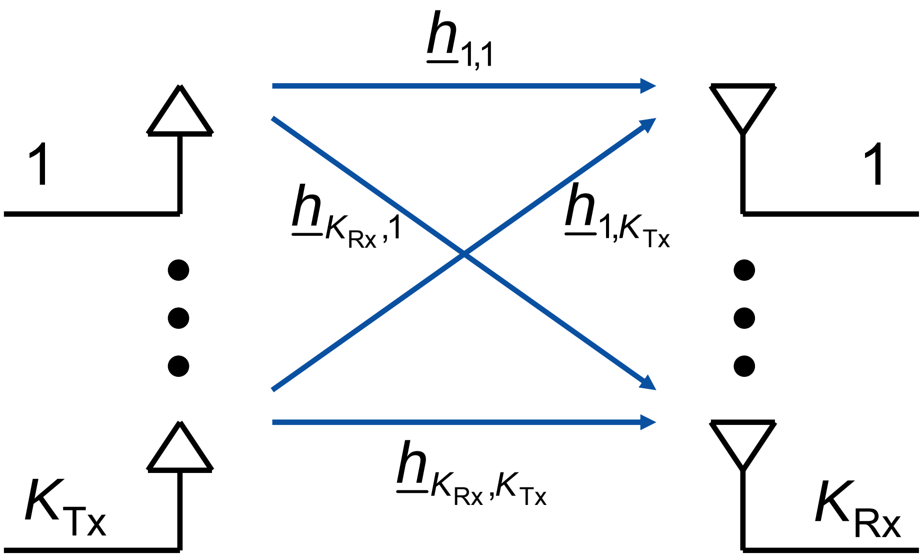
$$a(t-\tau) = \sqrt{2}\,\mathrm{Re}\left\{\underline{u}(t-\tau)\cdot \mathrm{e}^{\mathrm{j}2\pi f_0 (t-\tau)}\right\}$$

$$= \sqrt{2}\,\mathrm{Re}\left\{ \underbrace{\underline{u}(t-\tau)\cdot \mathrm{e}^{-\mathrm{j}2\pi f_0 \tau}}_{\text{lowpass equivalent of } a(t-\tau)} \cdot \mathrm{e}^{\mathrm{j}2\pi f_0 t}\right\}$$

$$\approx \sqrt{2}\,\mathrm{Re}\left\{\underline{u}(t)\cdot \mathrm{e}^{-\mathrm{j}2\pi f_0 \tau}\cdot \mathrm{e}^{\mathrm{j}2\pi f_0 t}\right\}$$

$\Rightarrow$ Small time shifts τ correspond to phase rotations by $\mathrm{e}^{-\mathrm{j}2\pi f_0 \tau}$ in equivalent lowpass domain.

MIMO System, Single Tap Channel



SISO subsystem:

$$\underline{e}_{k_{Rx}} = \underline{h}_{k_{Rx},k_{Tx}} \cdot \underline{s}_{k_{Tx}}$$

MIMO system:

$$\underbrace{\begin{pmatrix} \underline{e}_1 \\ \vdots \\ \underline{e}_{K_{Rx}} \end{pmatrix}}_{\underline{\mathbf{e}}} = \underbrace{\begin{pmatrix} \underline{h}_{1,1} & \cdots & \underline{h}_{1,K_{Tx}} \\ \vdots & & \vdots \\ \underline{h}_{K_{Rx},1} & \cdots & \underline{h}_{K_{Rx},K_{Tx}} \end{pmatrix}}_{\underline{\mathbf{H}}} \cdot \underbrace{\begin{pmatrix} \underline{s}_1 \\ \vdots \\ \underline{s}_{K_{Tx}} \end{pmatrix}}_{\underline{\mathbf{s}}}$$

channel matrix $\underline{\mathbf{H}}$ is a $K_{Rx} \times K_{Tx}$ matrix

White Multivariate Gaussian Noise

$$p_n(\underline{\mathbf{n}}) = p\left(\operatorname{Re}\{\underline{n}_0\}\right)\cdot p\left(\operatorname{Im}\{\underline{n}_0\}\right)$$

$$\vdots$$

$$\cdot p\left(\operatorname{Re}\{\underline{n}_{M-1}\}\right)\cdot p\left(\operatorname{Im}\{\underline{n}_{M-1}\}\right)$$

$$= \frac{1}{\sqrt{\pi\sigma^2}} e^{-\frac{\operatorname{Re}\{\underline{n}_0\}^2}{\sigma^2}} \cdot \frac{1}{\sqrt{\pi\sigma^2}} e^{-\frac{\operatorname{Im}\{\underline{n}_0\}^2}{\sigma^2}}$$

$$\vdots$$

$$\cdot \frac{1}{\sqrt{\pi\sigma^2}} e^{-\frac{\operatorname{Re}\{\underline{n}_{M-1}\}^2}{\sigma^2}} \cdot \frac{1}{\sqrt{\pi\sigma^2}} e^{-\frac{\operatorname{Im}\{\underline{n}_{M-1}\}^2}{\sigma^2}}$$

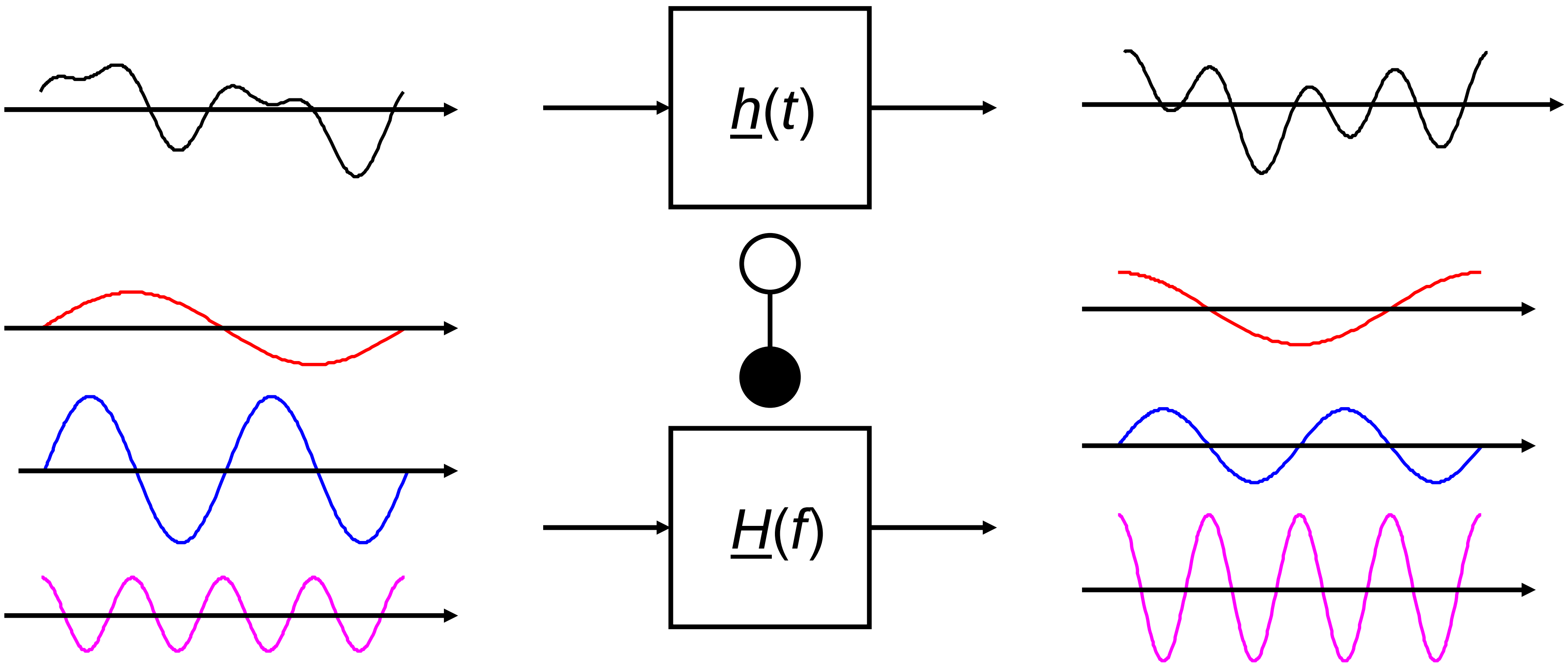
$$p_n(\underline{\mathbf{n}}) = \frac{1}{\left(\pi\sigma^2\right)^M} \cdot e^{-\frac{1}{\sigma^2}\underline{\mathbf{n}}^{*T}\underline{\mathbf{n}}}$$

notation: $\underline{n}_m \sim \mathbb{C}\mathcal{N}\left\{0, \sigma^2\right\}$

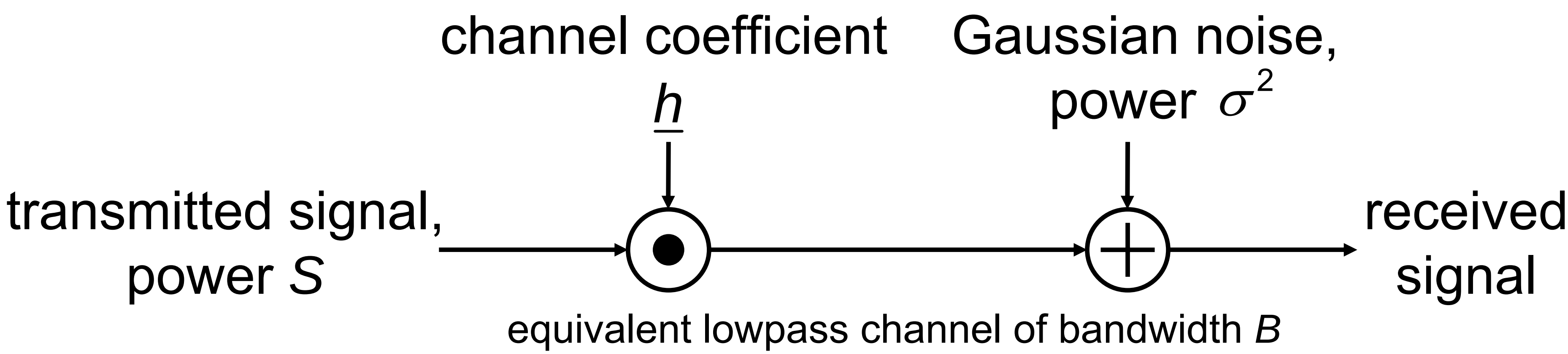
$$\underline{\mathbf{n}} \sim \mathbb{C}\mathcal{N}\left\{0, \sigma^2 \mathbf{I}\right\}$$

independent identically distributed, i.i.d.

Multi Carrier Transmission, OFDM



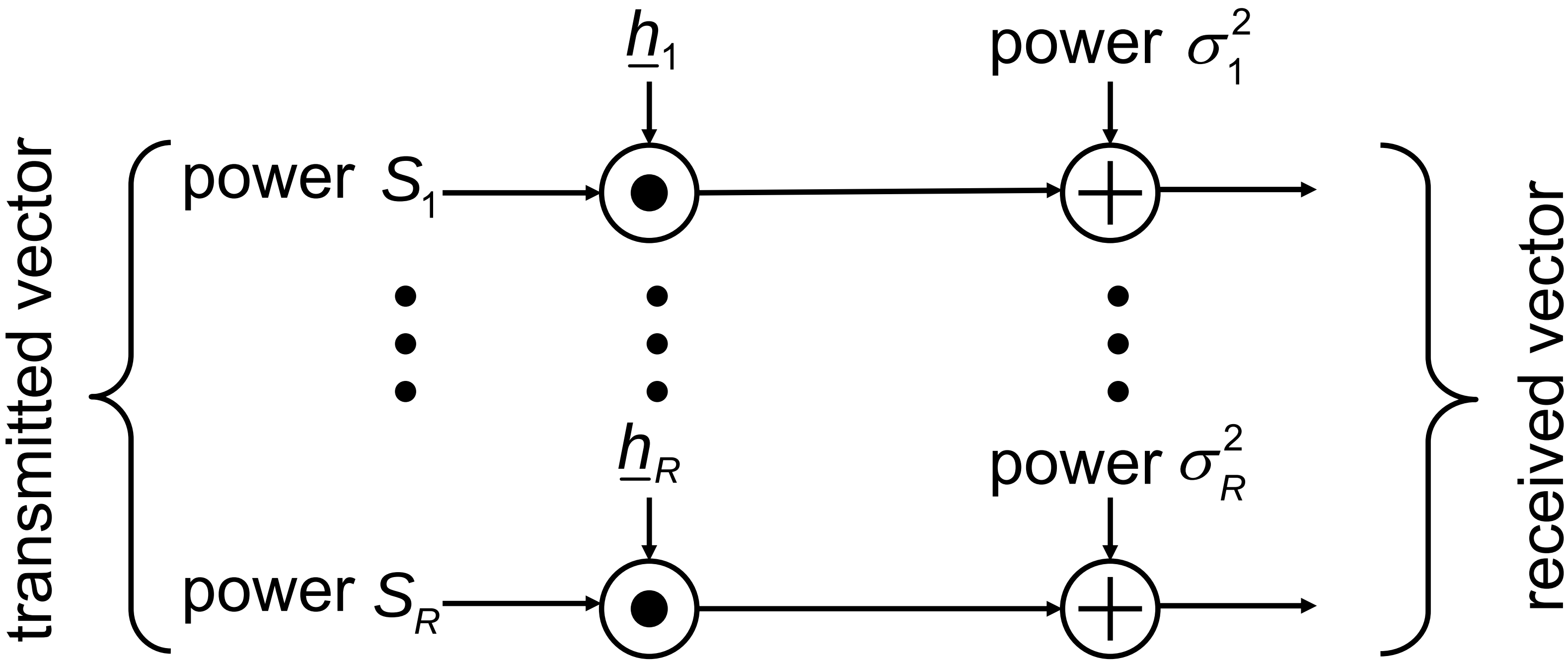
SISO Channel Capacity



Shannon (1948): *A Mathematical Theory of Communication*
per channel use (Nyquist rate):

$$C = \mathrm{ld}\left(1 + \frac{|\underline{h}|^2 S}{\sigma^2}\right), \quad [C] = \frac{\mathrm{bit}}{\mathrm{s} \cdot \mathrm{Hz}}$$

Uncoupled (R, R) MIMO Channel



total transmitted power: $S = \sum_{r=1}^{R} S_r$

total channel capacity: $C = \sum_{r=1}^{R} C_r = \sum_{r=1}^{R} \mathrm{ld}\left(1 + \frac{|\underline{h}_r|^2 S_r}{\sigma_r^2}\right)$

All R parallel channels get the same transmitted power:

$$S_r = \frac{S}{R}$$

$$C = \sum_{r=1}^{R} \mathrm{ld}\left(1 + \frac{\left|\underline{h}_r\right|^2}{\sigma_r^2} \cdot \frac{S}{R}\right) = \mathrm{ld} \prod_{r=1}^{R} \left(1 + \frac{\left|\underline{h}_r\right|^2}{\sigma_r^2} \cdot \frac{S}{R}\right)$$

Optimization Task

Question: How large is the total channel capacity C for limited total transmitted power S and how can it be achieved?

Idea: Allocate the total transmitted power S in a smart way to the R parallel channels!

maximize $C = \sum_{r=1}^{R} \mathrm{ld}\left(1 + \frac{|\underline{h}_r|^2 S_r}{\sigma_r^2}\right)$

subject to the constraint $S = \sum_{r=1}^{R} S_r$

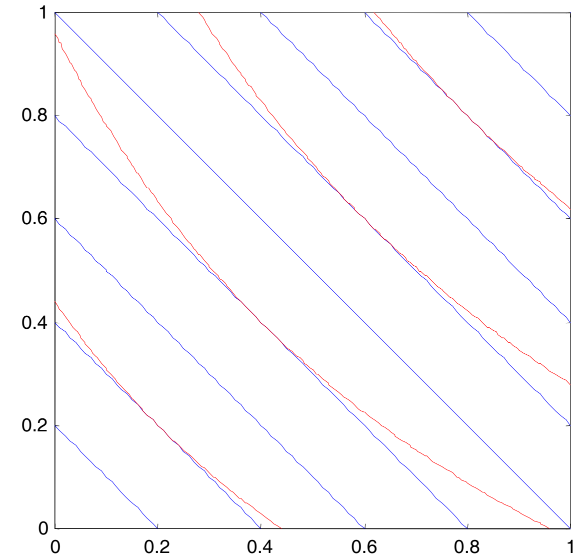
$\Rightarrow$ method of Lagrangian multipliers

Optimization (1)

definition: $\mathbf{S} = \begin{pmatrix} S_1 \\ \vdots \\ S_R \end{pmatrix}$

$$\text{maximize } \mathrm{f}(\mathbf{S}) = \sum_{r=1}^{R} \mathrm{ld}\left(1 + \frac{|h_r|^2 S_r}{\sigma_r^2}\right)$$

$$\text{subject to } \mathrm{g}(\mathbf{S}) = \sum_{r=1}^{R} S_r - S = 0$$



R = 2; identical channels

f(**S**) = const., g(**S**) = const.

Lagrange: $\operatorname{grad} \mathrm{f}(\mathbf{S}) + \lambda \operatorname{grad} \mathrm{g}(\mathbf{S}) = \mathbf{0}$

$$\operatorname{grad} \mathrm{f}(\mathbf{S}) = \begin{pmatrix} \frac{\partial \mathrm{f}(\mathbf{S})}{\partial S_1} \\ \vdots \\ \frac{\partial \mathrm{f}(\mathbf{S})}{\partial S_R} \end{pmatrix}, \quad \operatorname{grad} \mathrm{g}(\mathbf{S}) = \begin{pmatrix} \frac{\partial \mathrm{g}(\mathbf{S})}{\partial S_1} \\ \vdots \\ \frac{\partial \mathrm{g}(\mathbf{S})}{\partial S_R} \end{pmatrix}$$

Optimization (2)

with
$$\frac{\partial f(\mathbf{S})}{\partial S_r} = \frac{1}{\ln 2} \cdot \frac{1}{1 + \frac{|h_r|^2 \cdot S_r}{\sigma_r^2}} \cdot \frac{|h_r|^2}{\sigma_r^2} = \frac{1}{\ln 2} \cdot \frac{1}{\frac{\sigma_r^2}{|h_r|^2} + S_r}$$

$$\frac{\partial g(\mathbf{S})}{\partial S_1} = 1$$

$$\Rightarrow \operatorname{grad} f(\mathbf{S}) + \lambda \operatorname{grad} g(\mathbf{S}) = \frac{1}{\ln 2} \begin{pmatrix} \frac{1}{\frac{\sigma_1^2}{|\underline{h}_1|^2} + S_1} \\ \vdots \\ \frac{1}{\frac{\sigma_R^2}{|\underline{h}_R|^2} + S_R} \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \mathbf{0}$$

Optimization (3)

$$\Rightarrow \frac{\sigma_r^2}{\left|\underline{h}_r\right|^2} + S_r = -\frac{1}{\lambda \ln 2} = S_\mathrm{W} = \mathrm{const}$$

$$\Rightarrow S_r = S_\mathrm{W} - \frac{\sigma_r^2}{\left|\underline{h}_r\right|^2}$$

attention: $S_r \geq 0$

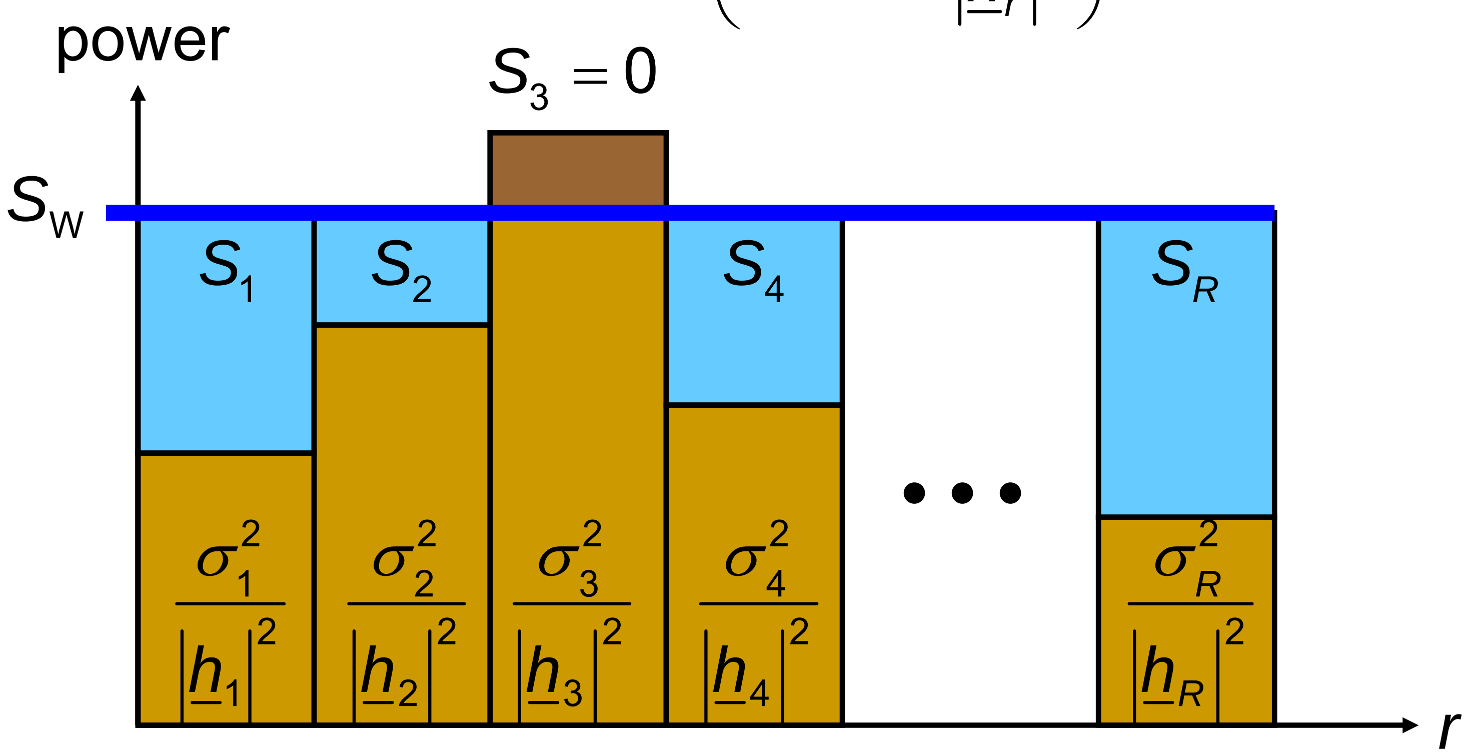
$$\Rightarrow S_r = \max\left(0, S_\mathrm{W} - \frac{\sigma_r^2}{\left|\underline{h}_r\right|^2}\right)$$

where S_W is chosen such that

$$\sum_{r=1}^{R} S_r = S$$

Waterfilling

$$S_r = \max\left(0, S_W - \frac{\sigma_r^2}{|\underline{h}_r|^2}\right) \qquad S = \sum_{r=1}^{R} S_r$$



Holsinger (1964): *Digital communication over fixed time-continous channels with memory - with special application to telephone channels* requires transmitter side channel state information

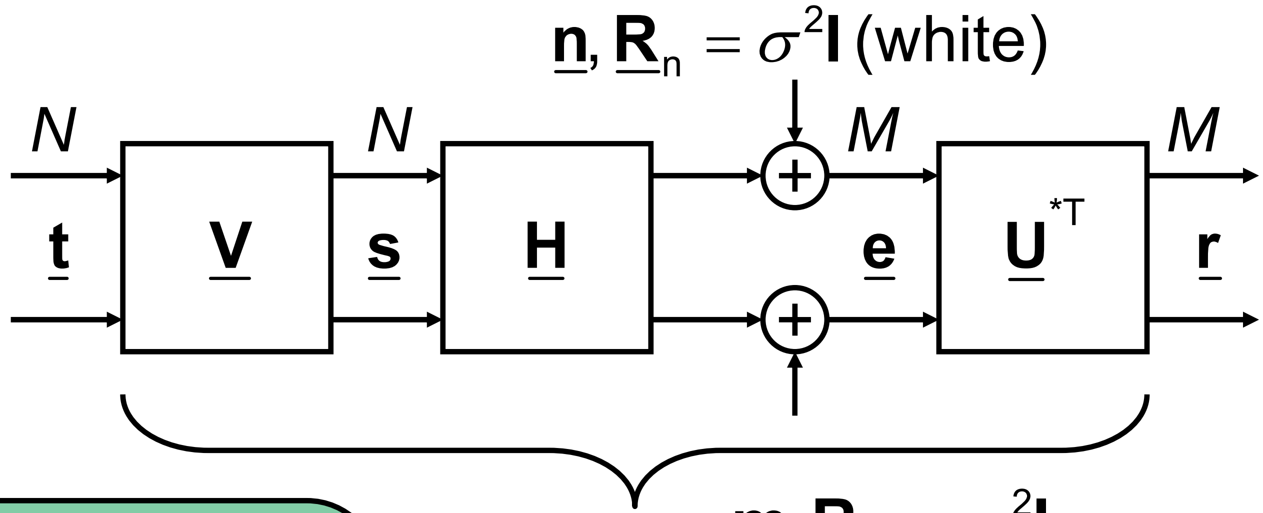
$$C = \sum_{r=1}^{R} \mathrm{ld}\left(1 + \frac{|\underline{h}_r|^2}{\sigma_r^2} \underbrace{\max\left(0, S_{\mathrm{W}} - \frac{\sigma_r^2}{|\underline{h}_r|^2}\right)}_{S_r}\right) = \sum_{r=1}^{R} \max\left(0, \mathrm{ld}\left(\frac{|\underline{h}_r|^2 S_{\mathrm{W}}}{\sigma_r^2}\right)\right)$$

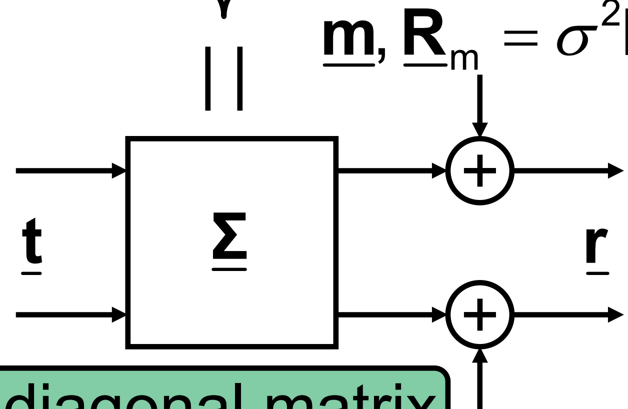
special case: all R channels used

$$S_r = S_{\mathrm{W}} - \frac{\sigma_r^2}{|\underline{h}_r|^2} \Rightarrow \sum_{r=1}^{R} \left(S_W - \frac{\sigma_r^2}{|\underline{h}_r|^2}\right) = S \Rightarrow S_{\mathrm{W}} = \frac{S}{R} + \frac{1}{R} \sum_{r=1}^{R} \frac{\sigma_r^2}{|\underline{h}_r|^2}$$

$$\Rightarrow C = \sum_{r=1}^{R} \mathrm{ld}\left(\frac{|\underline{h}_r|^2 S_{\mathrm{w}}}{\sigma_r^2}\right) = \mathrm{ld} \prod_{r=1}^{R} \frac{|\underline{h}_r|^2 S_{\mathrm{w}}}{\sigma_r^2}$$

Challenge





$\underline{\mathbf{s}}$ should have same power as $\underline{\mathbf{t}}$:

$$\underline{\mathbf{s}}^{*T}\underline{\mathbf{s}} = \underline{\mathbf{t}}^{*T}\underline{\mathbf{V}}^{*T}\underline{\mathbf{V}}\underline{\mathbf{t}}$$

$$\rightarrow \underline{\mathbf{V}}^{*T}\underline{\mathbf{V}} = \mathbf{I} \text{ (unitary)}$$

diagonal matrix

$\underline{\mathbf{m}}$ should be white:

$$\underline{\mathbf{R}}_m = \sigma^2 \underline{\mathbf{U}}^{*T}\underline{\mathbf{U}}$$

$$\rightarrow \underline{\mathbf{U}}^{*T}\underline{\mathbf{U}} = \mathbf{I} \text{ (unitary)}$$

find unitary matrices $\underline{\mathbf{U}}$, $\underline{\mathbf{V}}$ and a diagonal matrix $\underline{\mathbf{\Sigma}}$ such that

$$\underline{\mathbf{\Sigma}} = \underline{\mathbf{U}}^{*T} \cdot \underline{\mathbf{H}} \cdot \underline{\mathbf{V}} \Leftrightarrow \underline{\mathbf{H}} = \underline{\mathbf{U}} \cdot \underline{\mathbf{\Sigma}} \cdot \underline{\mathbf{V}}^{*T}$$

Singular Value Decomposition (1)

Singular Value Decomposition Theorem (Eckart & Young: 1939):

For every $M \times N$ matrix $\underline{\mathbf{H}}$ there are two unitary matrices $\underline{\mathbf{U}}$ and $\underline{\mathbf{V}}$, such that

$$\boldsymbol{\Sigma} = \underline{\mathbf{U}}^{*\mathrm{T}} \cdot \underline{\mathbf{H}} \cdot \underline{\mathbf{V}}$$

is a $M \times N$ diagonal matrix with nonnegative real diagonal elements.

Singular Value Decomposition (2)

$$\boldsymbol{\Sigma} = \underline{\mathbf{U}}^{*T} \cdot \underline{\mathbf{H}} \cdot \underline{\mathbf{V}} \Leftrightarrow \underline{\mathbf{H}} = \underline{\mathbf{U}} \cdot \boldsymbol{\Sigma} \cdot \underline{\mathbf{V}}^{*T}$$

$\underline{\mathbf{U}}$: unitary $M \times M$ matrix, columns are named left singular vectors and correspond to eigenvectors of $\underline{\mathbf{H}}\underline{\mathbf{H}}^{*T} = \underline{\mathbf{U}}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{*T}\underline{\mathbf{U}}^{*T}$

$\underline{\mathbf{V}}$: unitary $N \times N$ matrix, columns are named right singular vectors and correspond to eigenvectors of $\underline{\mathbf{H}}^{*T}\underline{\mathbf{H}} = \underline{\mathbf{V}}\boldsymbol{\Sigma}^{*T}\boldsymbol{\Sigma}\underline{\mathbf{V}}^{*T}$

Σ: $M \times N$ diagonal matrix, diagonal elements are named singular values and correspond to the square roots of the eigenvalues λ_q of $\underline{\mathbf{H}}\underline{\mathbf{H}}^{*T}$ or $\underline{\mathbf{H}}^{*T}\underline{\mathbf{H}}$

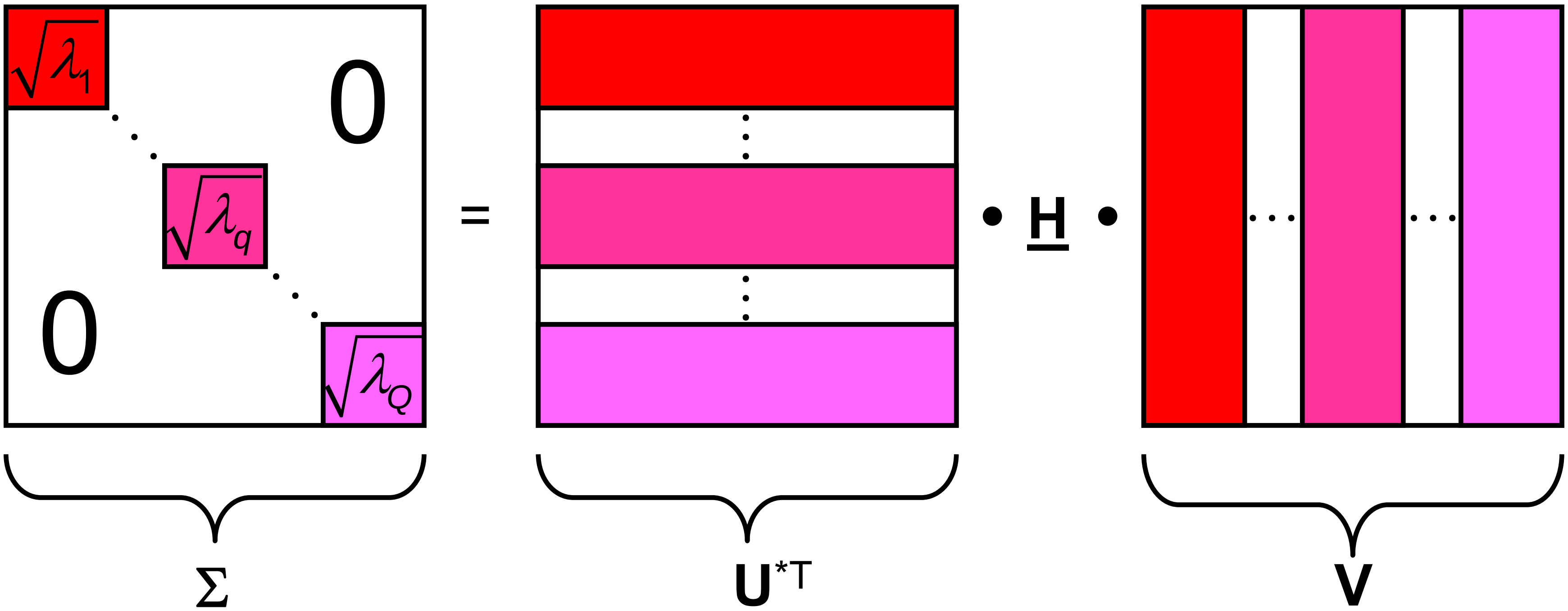
$\underline{\mathbf{H}}\underline{\mathbf{H}}^{*T}$ Grammian of the row vectors

$\underline{\mathbf{H}}^{*T}\underline{\mathbf{H}}$ Grammian of the column vectors

Matrix Structures (1)

$$\mathbf{\Sigma} = \underline{\mathbf{U}}^{*T} \cdot \underline{\mathbf{H}} \cdot \underline{\mathbf{V}} \Leftrightarrow \underline{\mathbf{H}} = \underline{\mathbf{U}} \cdot \mathbf{\Sigma} \cdot \underline{\mathbf{V}}^{*T}$$

example: $M = N$



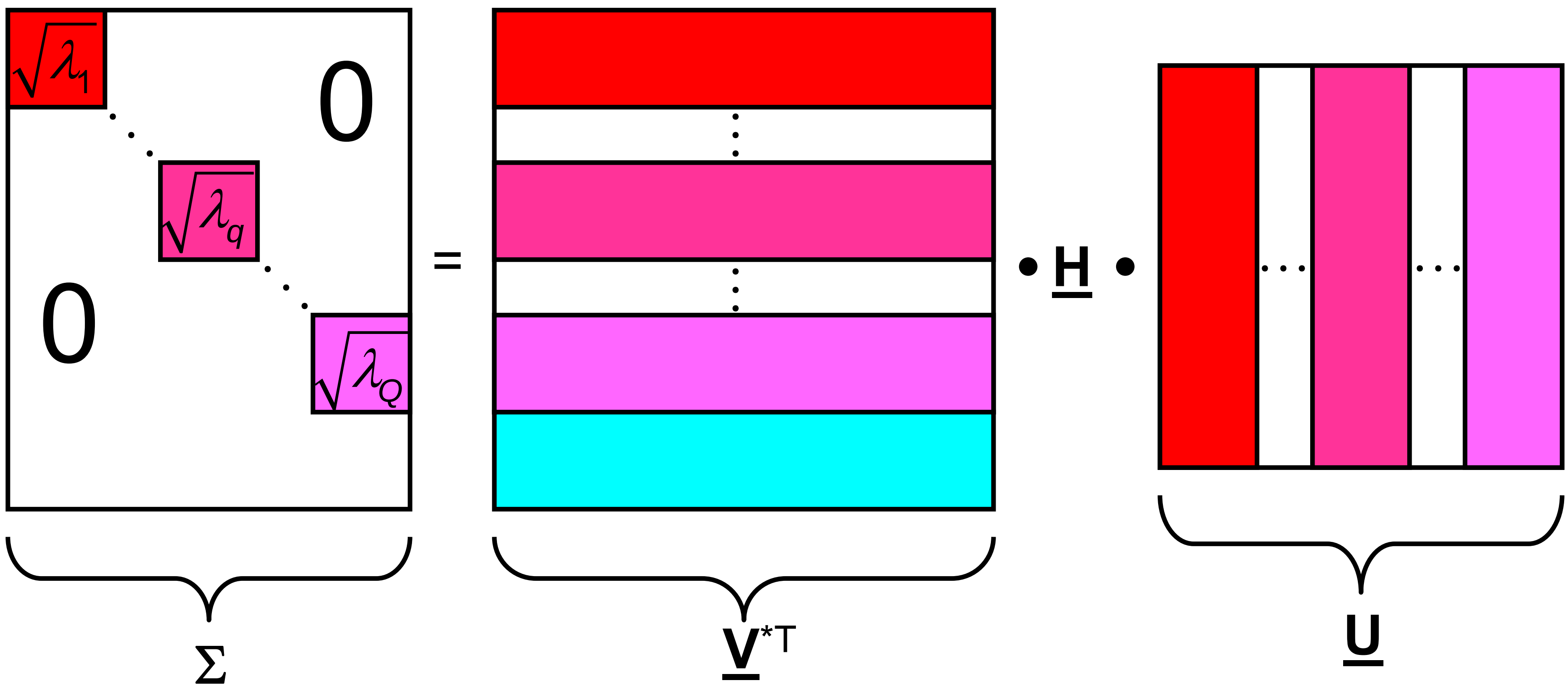
$$\sqrt{\lambda_1} \geq \sqrt{\lambda_2} \geq \ldots \geq \sqrt{\lambda_R} > \sqrt{\lambda_{R+1}} = \ldots = \sqrt{\lambda_Q} = 0$$

rank of the channel: $R = \mathrm{rank}\left(\underline{\mathbf{H}}\right) \leq Q = \min\left(N, M\right)$

Matrix Structures (2)

$$\mathbf{\Sigma} = \underline{\mathbf{U}}^{*T} \cdot \underline{\mathbf{H}} \cdot \underline{\mathbf{V}} \Leftrightarrow \underline{\mathbf{H}} = \underline{\mathbf{U}} \cdot \mathbf{\Sigma} \cdot \underline{\mathbf{V}}^{*T}$$

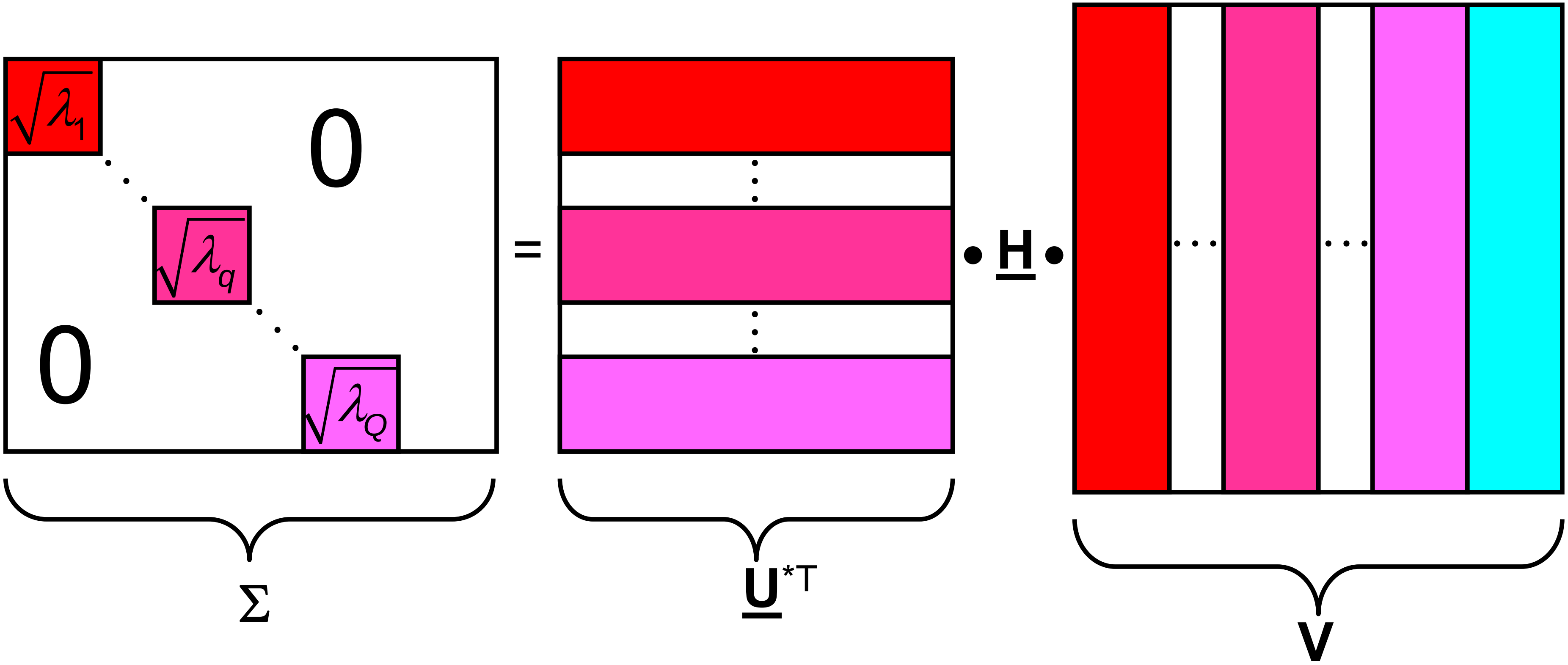
example: $M > N$



Matrix Structures (3)

$$\mathbf{\Sigma} = \underline{\mathbf{U}}^{*T} \cdot \underline{\mathbf{H}} \cdot \underline{\mathbf{V}} \Leftrightarrow \underline{\mathbf{H}} = \underline{\mathbf{U}} \cdot \mathbf{\Sigma} \cdot \underline{\mathbf{V}}^{*T}$$

example: $M < N$



without transmitter side channel state information
(Foschini, 1996):
all inputs with same power

$$C = \sum_{r=1}^{R} \mathrm{ld}\left(1 + \frac{\lambda_r}{\sigma^2}\frac{S}{N}\right) = \mathrm{ld}\prod_{r=1}^{R}\left(1 + \frac{\lambda_r}{\sigma^2}\frac{S}{N}\right)$$

$$= \mathrm{ld}\left(\det\left(\mathbf{I} + \frac{S}{N\sigma^2}\Sigma^{*T}\Sigma\right)\right)$$

$$= \mathrm{ld}\left(\det\left(\mathbf{I} + \frac{S}{N\sigma^2}\underline{\mathbf{V}}^{*T}\underline{\mathbf{H}}^{*T}\underline{\mathbf{U}}\underline{\mathbf{U}}^{*T}\underline{\mathbf{H}}\underline{\mathbf{V}}\right)\right) = \mathrm{ld}\left(\det\left(\underline{\mathbf{V}}^{*T}\left(\mathbf{I} + \frac{S}{N\sigma^2}\underline{\mathbf{H}}^{*T}\underline{\mathbf{H}}\right)\underline{\mathbf{V}}\right)\right)$$

$$= \mathrm{ld}\left(\det\left(\underline{\mathbf{V}}^{*T}\right)\cdot\det\left(\mathbf{I} + \frac{S}{N\sigma^2}\underline{\mathbf{H}}^{*T}\underline{\mathbf{H}}\right)\cdot\det\left(\underline{\mathbf{V}}\right)\right) = \mathrm{ld}\left(\det\left(\mathbf{I} + \frac{S}{N\sigma^2}\underline{\mathbf{H}}^{*T}\underline{\mathbf{H}}\right)\right)$$

$$= \mathrm{ld}\left(\det\left(\mathbf{I} + \frac{S}{N\sigma^2}\underline{\mathbf{H}}\underline{\mathbf{H}}^{*T}\right)\right)$$

with transmitter side channel state information (Telatar, 1995, 1999):

$$C = \sum_{r=1}^{R} \max\left(0, \operatorname{ld}\left(\frac{\lambda_r S_W}{\sigma^2}\right)\right)$$

S_W such that

$$S = \sum_{r=1}^{R} S_r = \sum_{r=1}^{R} \max\left(0, S_W - \frac{\sigma^2}{\lambda_r}\right)$$

Capacity of Stochastic Channels

- instantaneous channel capacity

$$C_{\text{inst}} = \begin{cases} \sum_{r=1}^{R} \max\left(0, \text{ld}\left(\frac{\lambda_r S_{\text{W}}}{\sigma^2}\right)\right) & \text{with TxCSI} \\ \sum_{r=1}^{R} \text{ld}\left(1 + \frac{\lambda_r S}{\sigma^2 N}\right) & \text{without TxCSI} \end{cases}$$

- complementary cumulative distribution function

$$\Pr\left\{C_{\text{inst}} > C\right\} = \int_{C}^{\infty} \text{p}\left(C_{\text{inst}}\right) \text{d}C_{\text{inst}}$$

- ergodic channel capacity

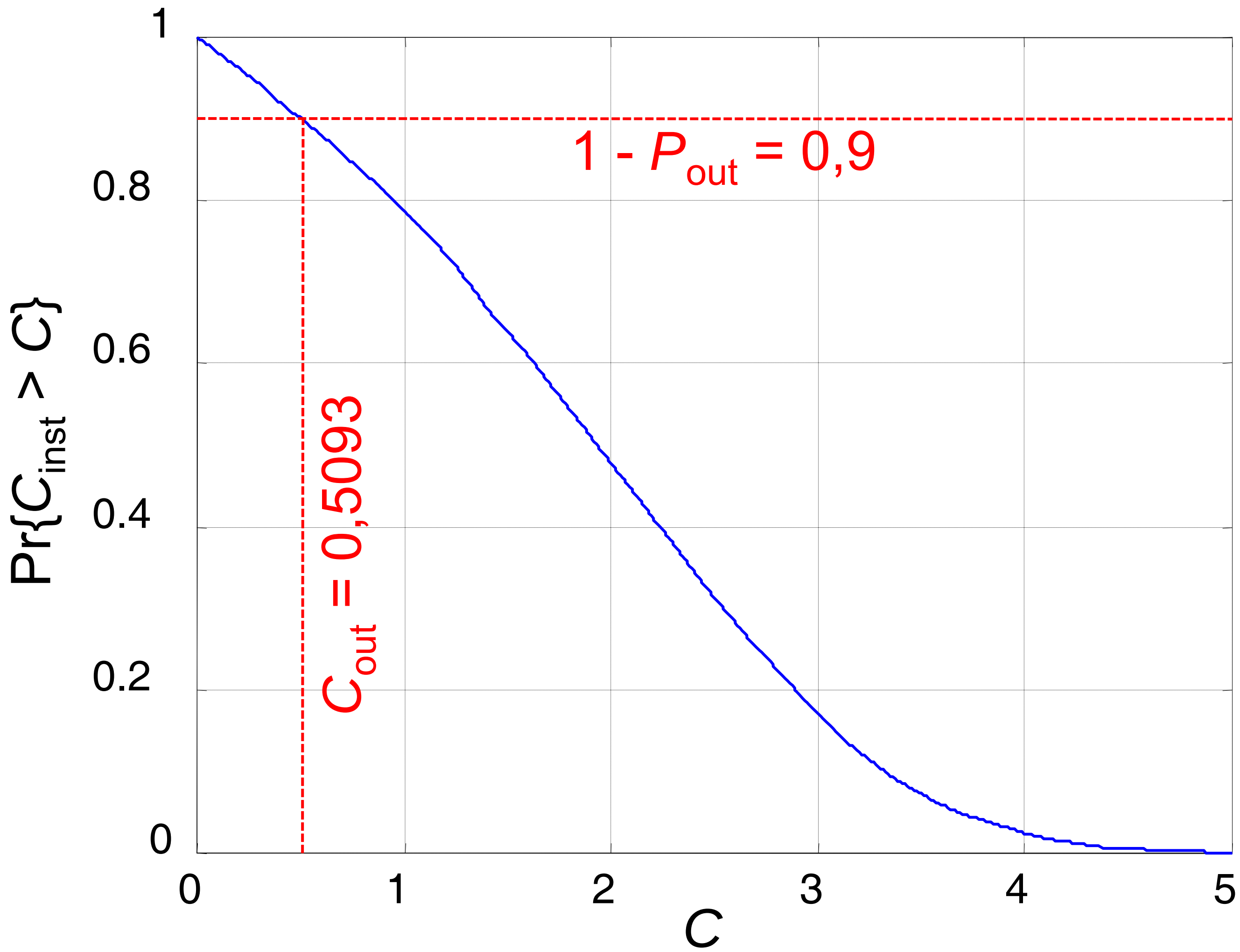
$$C_{\text{erg}} = \text{E}\left\{C_{\text{inst}}\right\}$$

- outage channel capacity, outage probability P_{out}

$$\Pr\left\{C_{\text{inst}} > C_{\text{out}}\right\} = 1 - P_{\text{out}}$$

Graphical Evaluation

Institut für Nachrichtentechnik

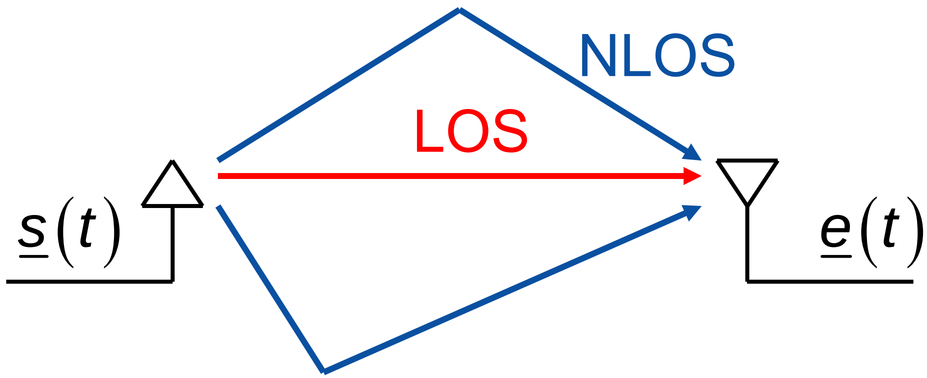


parameters:

- $N = M = 1$
- $S/\sigma^2 = 4$
- $E\{|\underline{h}|^2\} = 1$
- Rayleigh

$C_{erg} = 1{,}9415$

Deterministic SISO Channel Model



LOS: line of sight,
NLOS: non line of sight

time domain

- not band limited
$$\underline{\tilde{h}}(t) = \sum_{p=1}^{P} \underline{a}_p \cdot \delta\left(t - \tau_p\right)$$
- band limited
$$\underline{h}(t) = \sum_{p=1}^{P} \underline{a}_p \cdot B \cdot \mathrm{sinc}\left(B\left(t - \tau_p\right)\right)$$
- in general time dispersive, i.e., impulse response spread in time

frequency domain

- not band limited
$$\underline{\tilde{H}}(f) = \sum_{p=1}^{P} \underline{a}_p \cdot \mathrm{e}^{-\mathrm{j}2\pi f \tau_p}$$
- band limited
$$\underline{H}(f) = \sum_{p=1}^{P} \underline{a}_p \cdot \mathrm{e}^{-\mathrm{j}2\pi f \tau_p} \cdot \mathrm{rect}\left(\frac{f}{B}\right)$$
- in general frequency selective, i.e., frequency dependent transfer function

Single Tap Channel

time domain

- single tap channel

$$\left|\tau_p - \tau_{p'}\right| \ll \frac{1}{B} \text{ für alle } p, p'$$

- impulse response

$$\underline{h}(t) \approx \underline{h} \cdot B \cdot \operatorname{sinc}\left(B\left(t-\tau\right)\right)$$

frequency domain

- flat fading channel

$$\left|\underline{H}(f)\right| \approx \text{const}$$

- transfer function

$$\underline{H}(f) = \underline{h} \cdot \mathrm{e}^{-\mathrm{j}2\pi f \tau} \cdot \operatorname{rect}\left(\frac{f}{B}\right)$$

$$\underline{h} = \sum_{p=1}^{P} \underline{a}_p$$

neglect access delay τ

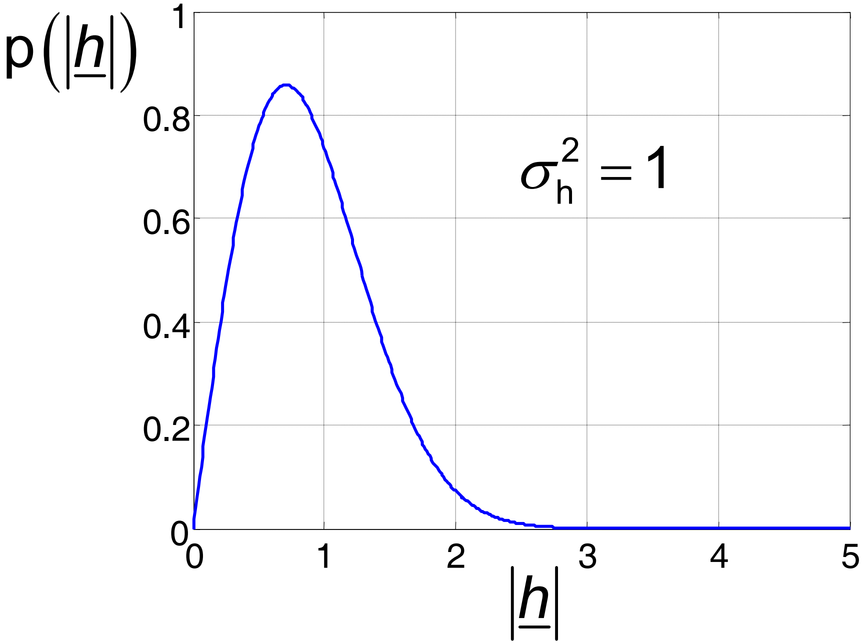
$$\Rightarrow \underline{h} = \sum_{p=1}^{P} \underline{a}_p$$

if the $\underline{a}_p$ are uncorrelated follows for $P \to \infty$ (central limit theorem):

$$\underline{h} \sim \mathbb{C}\mathcal{N}\left\{0, \sigma_h^2\right\}$$

$\Rightarrow |\underline{h}|$ is Rayleigh distributed:

$$\mathrm{p}(|\underline{h}|) = \begin{cases} \dfrac{2|\underline{h}|}{\sigma_h^2} \cdot e^{-\frac{|\underline{h}|^2}{\sigma_h^2}} & |\underline{h}| > 0 \\ 0 & \text{else} \end{cases}$$



Capacity of Rayleigh Channels

- average SNR

$$\gamma = \frac{\sigma_h^2 \cdot S}{\sigma^2}$$

- complementary cumulative distribution function

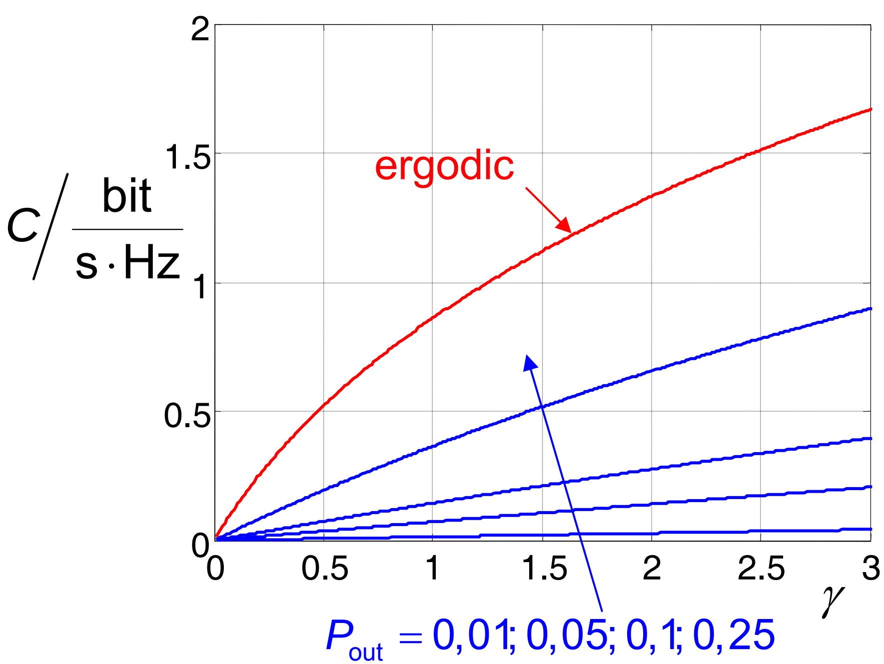
$$\Pr\{C_{\text{inst}} > C\} = \begin{cases} e^{-\frac{2^C - 1}{\gamma}} & C \geq 0 \\ 1 & \text{else} \end{cases}$$

- outage channel capacity

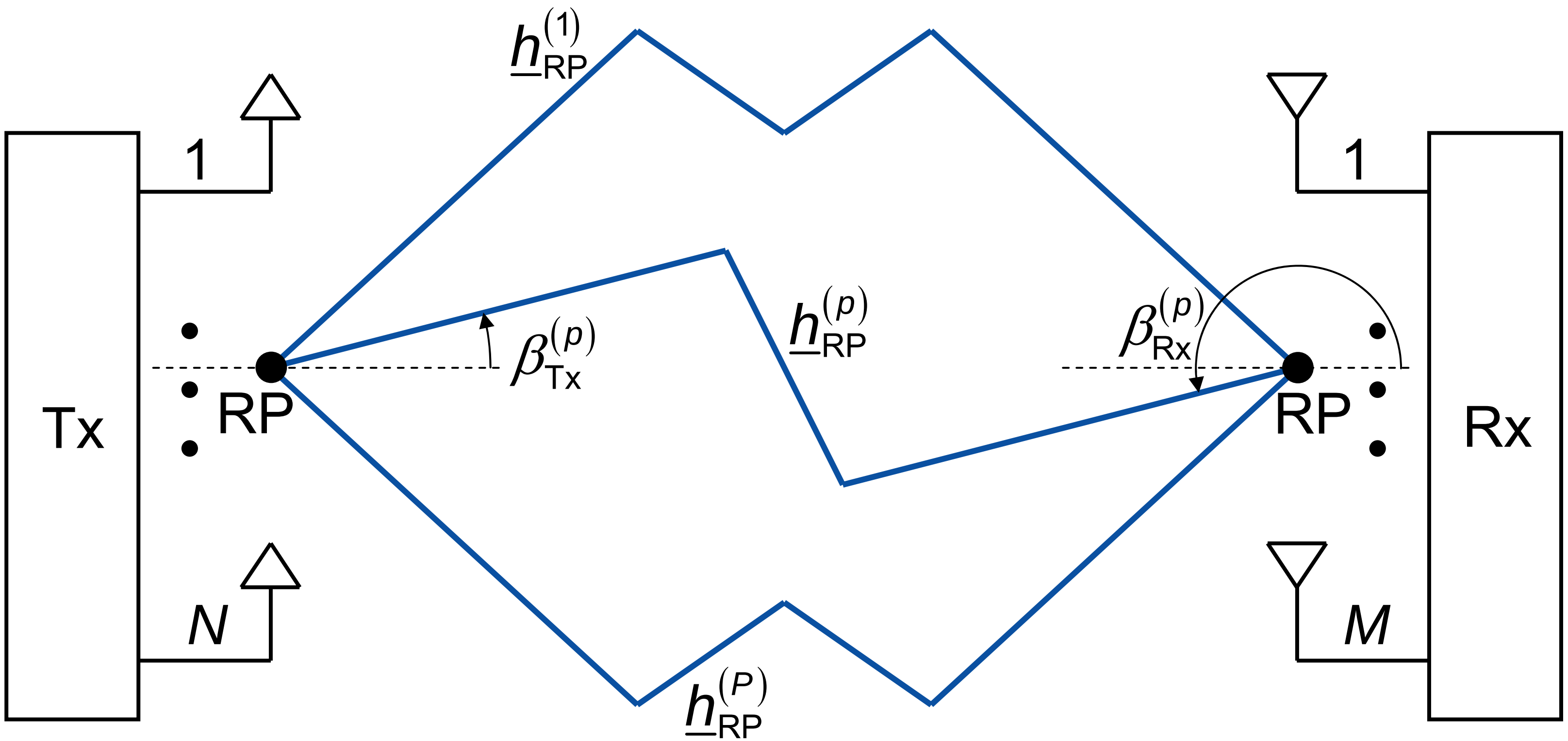
$$C_{\text{out}} = \text{ld}\left(1 - \gamma \cdot \ln\left(1 - P_{\text{out}}\right)\right)$$

- ergodic channel capacity

$$C_{\text{erg}} = \frac{e^{\frac{1}{\gamma}}}{\ln 2} \cdot \text{E}_1\left(\frac{1}{\gamma}\right)$$

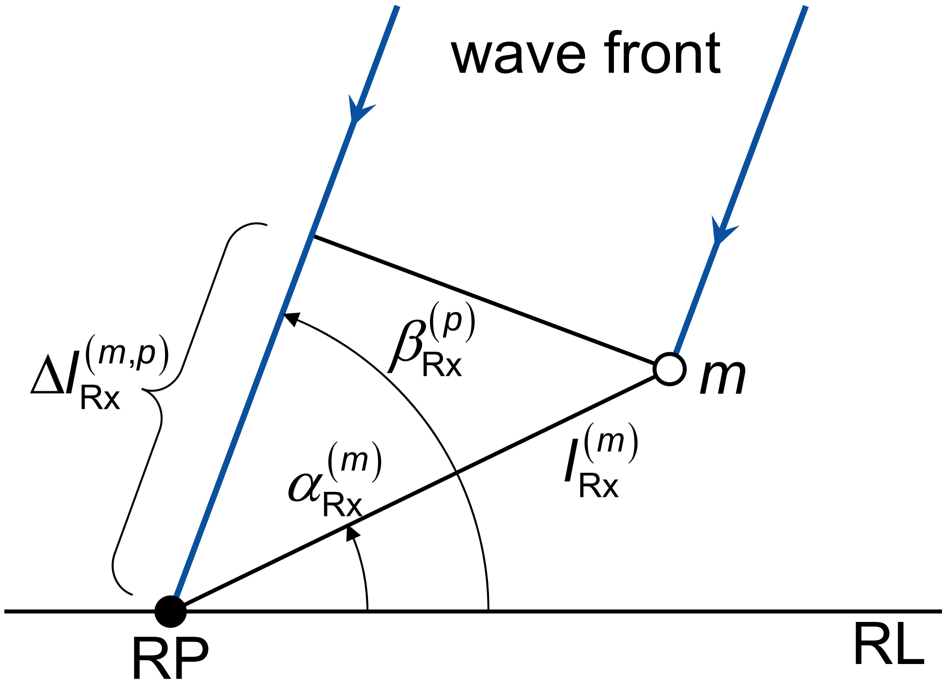


here: micro architectures, antenna arrays



- direction of departure, DOD: $\beta_{\mathrm{Tx}}^{(p)}$
- direction of arrival, DOA: $\beta_{\mathrm{Rx}}^{(p)}$
- directional channel coefficient: $\underline{h}_{\mathrm{RP}}^{(p)}$

Steering Factors and Steering Vector



$$\Delta l_{\text{Rx}}^{(m,p)} = l_{\text{Rx}}^{(m)} \cdot \cos\left(\beta_{\text{Rx}}^{(p)} - \alpha_{\text{Rx}}^{(m)}\right)$$

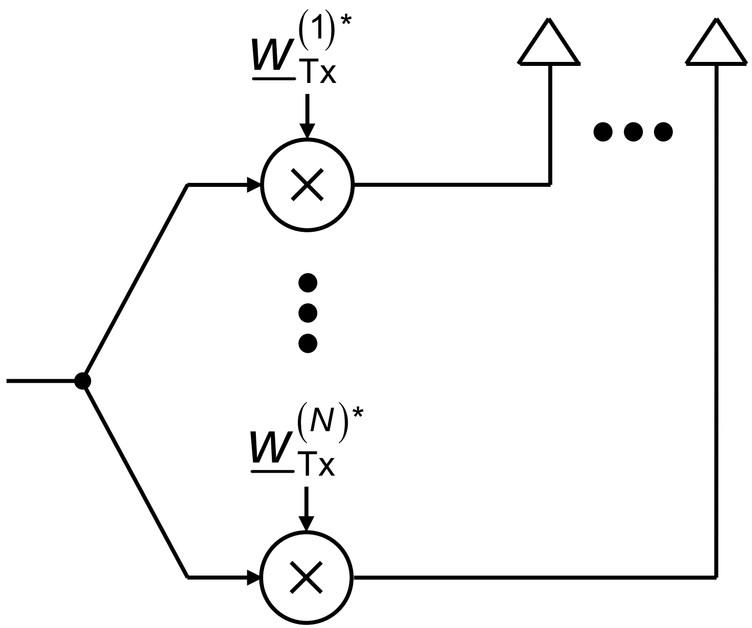
$$\varphi_{\text{Rx}}^{(m,p)} = \frac{2\pi}{\lambda} \Delta l_{\text{Rx}}^{(m,p)}$$

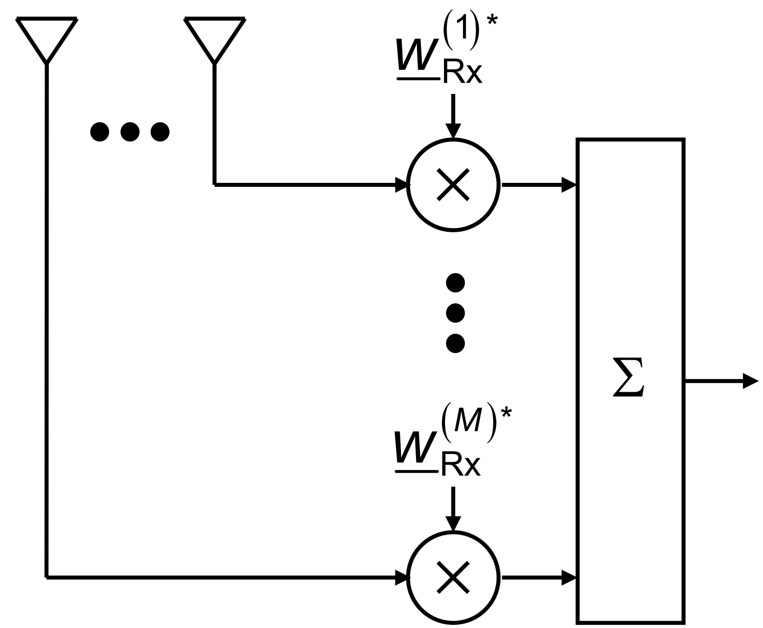
$$\underline{a}_{\text{Rx}}^{(m,p)} = \exp\left(\mathrm{j}\,\varphi_{\text{Rx}}^{(m,p)}\right)$$

$$\underline{\mathbf{a}}_{\text{Rx}}^{(p)} = \left(\underline{a}_{\text{Rx}}^{(1,p)} \ldots \underline{a}_{\text{Rx}}^{(M,p)}\right)^{\mathrm{T}}$$

due to reciprocity dual results hold for transmitter side

Weighting Network





transmitter side weighting vector:

$$\underline{\mathbf{w}}^*_{\mathrm{Tx}} = \left(\underline{w}^{(1)*}_{\mathrm{Tx}} \ldots \underline{w}^{(N)*}_{\mathrm{Tx}} \right)^{\mathrm{T}}$$

$$\left\| \underline{\mathbf{w}}^*_{\mathrm{Tx}} \right\|^2 = 1$$

receiver side weighting vector:

$$\underline{\mathbf{w}}^*_{\mathrm{Rx}} = \left(\underline{w}^{(1)*}_{\mathrm{Rx}} \ldots \underline{w}^{(M)*}_{\mathrm{Rx}} \right)^{\mathrm{T}}$$

$$\left\| \underline{\mathbf{w}}^*_{\mathrm{Rx}} \right\|^2 = 1$$

Antenna Gain

$$\underline{e}^{(m)} = \underline{a}_{\mathrm{Rx}}^{(m)} \cdot \underline{e}_{\mathrm{RP}}$$

$$\underline{e} = \sum_{m=1}^{M} \underline{w}_{\mathrm{Rx}}^{(m)*} \cdot \underline{e}^{(m)}$$

$$= \sum_{m=1}^{M} \underline{w}_{\mathrm{Rx}}^{(m)*} \cdot \underline{a}_{\mathrm{Rx}}^{(m)} \cdot \underline{e}_{\mathrm{RP}}$$

$$\underline{e} = \underline{e}_{\mathrm{RP}} \cdot \underbrace{\underline{\mathbf{w}}_{\mathrm{Rx}}^{*\mathrm{T}} \cdot \underline{\mathbf{a}}_{\mathrm{Rx}}}_{\text{skalarer Faktor}}$$

antenna gain:

$$g_{\mathrm{Rx}} = \left| \underline{\mathbf{w}}_{\mathrm{Rx}}^{*\mathrm{T}} \cdot \underline{\mathbf{a}}_{\mathrm{Rx}} \right|^2 = \underline{\mathbf{w}}_{\mathrm{Rx}}^{*\mathrm{T}} \cdot \underline{\mathbf{a}}_{\mathrm{Rx}} \cdot \underline{\mathbf{a}}_{\mathrm{Rx}}^{*\mathrm{T}} \cdot \underline{\mathbf{w}}_{\mathrm{Rx}}$$

due to reciprocity dual results hold for transmitter antennas

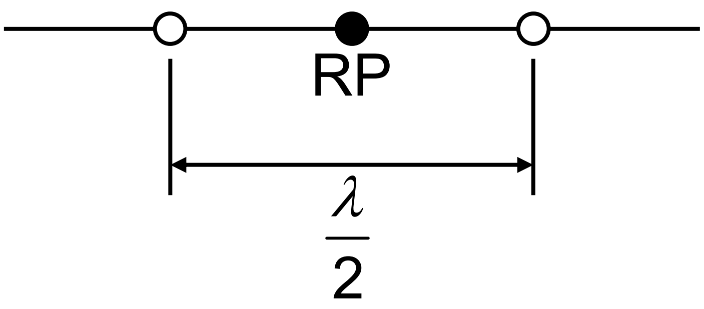
Antenna Diagram

consider antenna gain as a function of the DOA

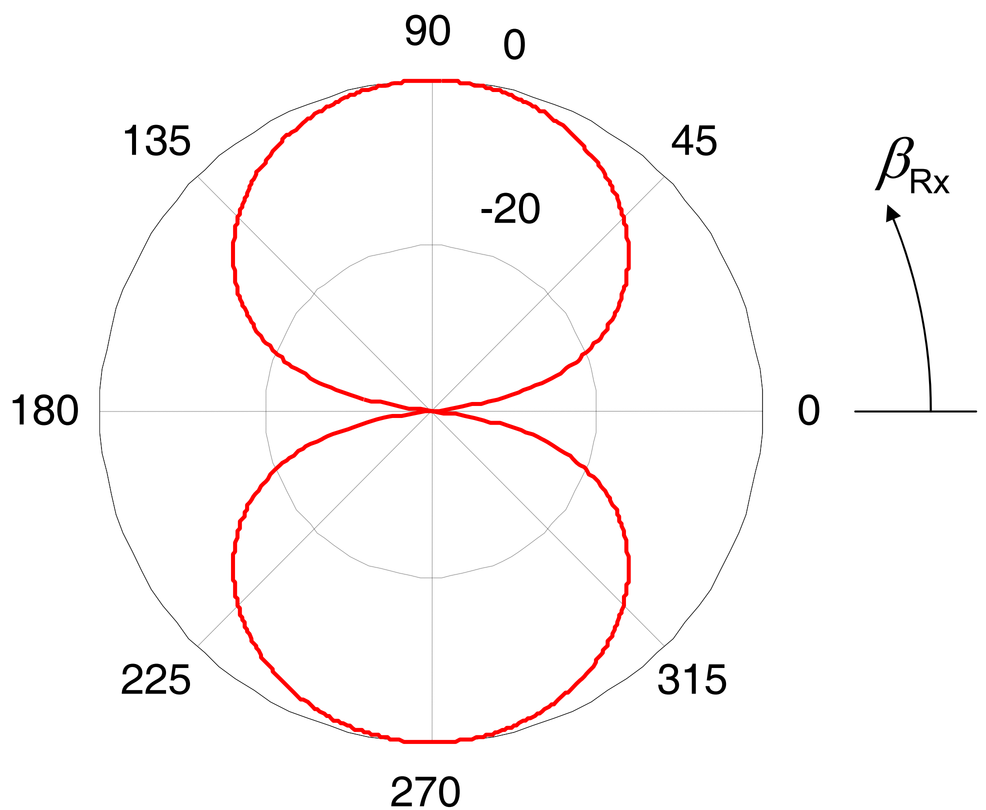
example:

$K_{\mathrm{Rx}} = 2$

$$\underline{w}_{\mathrm{Rx}}^{(1)*} = \underline{w}_{\mathrm{Rx}}^{(2)*} = \frac{1}{\sqrt{2}}$$



$$10 \cdot \log\left(\frac{g_{\mathrm{Rx}}(\beta_{\mathrm{Rx}})}{\max\left(g_{\mathrm{Rx}}(\beta_{\mathrm{Rx}})\right)}\right) \Big/ \mathrm{dB}$$



Conventional Beam Forming

maximize antenna gain

Schwarz inequality

$$g_{\mathrm{Rx}} = \left| \underline{\mathbf{w}}_{\mathrm{Rx}}^{*\mathrm{T}} \cdot \underline{\mathbf{a}}_{\mathrm{Rx}} \right|^2 \leq \left\| \underline{\mathbf{w}}_{\mathrm{Rx}}^{*\mathrm{T}} \right\|^2 \cdot \left\| \underline{\mathbf{a}}_{\mathrm{Rx}} \right\|^2$$

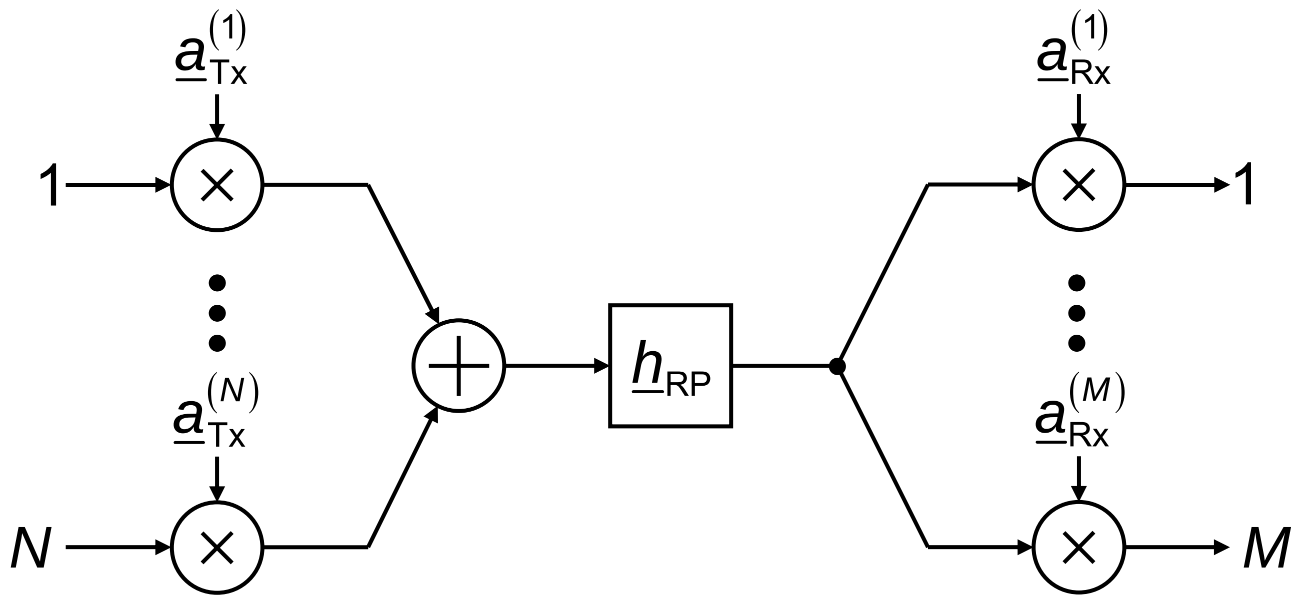
equality for

$$\underline{\mathbf{w}}_{\mathrm{Rx}} \sim \underline{\mathbf{a}}_{\mathrm{Rx}}$$

$\Rightarrow$ choose $\underline{\mathbf{w}}_{\mathrm{Rx}} = \frac{1}{\sqrt{M}} \underline{\mathbf{a}}_{\mathrm{Rx}}$ corresponds to maximal ratio combining, matched filtering

due to reciprocity dual results hold for transmitter side

Single Path Channel Model



SISO subsystem

- directional channel coefficient
  $\underline{h}_{\text{RP}}$
- spatial channel coefficient
  $\underline{h}^{(m,n)} = \underline{a}_{\text{Rx}}^{(m)} \cdot \underline{h}_{\text{RP}} \cdot \underline{a}_{\text{Tx}}^{(n)}$

MIMO system

- total channel matrix
  $\underline{\mathbf{H}} = \underline{\mathbf{a}}_{\text{Rx}} \cdot \underline{h}_{\text{RP}} \cdot \underline{\mathbf{a}}_{\text{Tx}}^{\text{T}}$

Singular Value Decomposition

$$\underline{\mathbf{U}} \cdot \mathbf{\Sigma} \cdot \underline{\mathbf{V}}^{*\mathrm{T}} = \left(\frac{1}{\sqrt{M}} \underline{\mathbf{a}}_{\mathrm{Rx}} \underbrace{\cdots\cdots}_{\text{orthonormal columns}} \right) \cdot e^{\mathrm{j}\arg\left(\underline{h}_{\mathrm{RP}}\right)}$$

$$\cdot \sqrt{MN} \begin{pmatrix} \left|\underline{h}_{\mathrm{RP}}\right| & 0 & \cdots & 0 \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 0 \end{pmatrix}$$

$$\cdot \begin{pmatrix} \frac{1}{\sqrt{N}} \underline{\mathbf{a}}_{\mathrm{Tx}}^{\mathrm{T}} \\ \left.\vdots\right\} \text{orthonormal rows} \end{pmatrix}$$

Capacity

- with transmitter side channel state information

$$C = \sum_{r=1}^{R} \max\left(0, \mathrm{ld}\left(\frac{\lambda_r S_\mathrm{W}}{\sigma^2}\right)\right) = \mathrm{ld}\left(1 + MN\frac{\left|\underline{h}_\mathrm{RP}\right|^2 S}{\sigma^2}\right)$$

$\Rightarrow$ SNR gain due to transmitter and receiver side beam forming

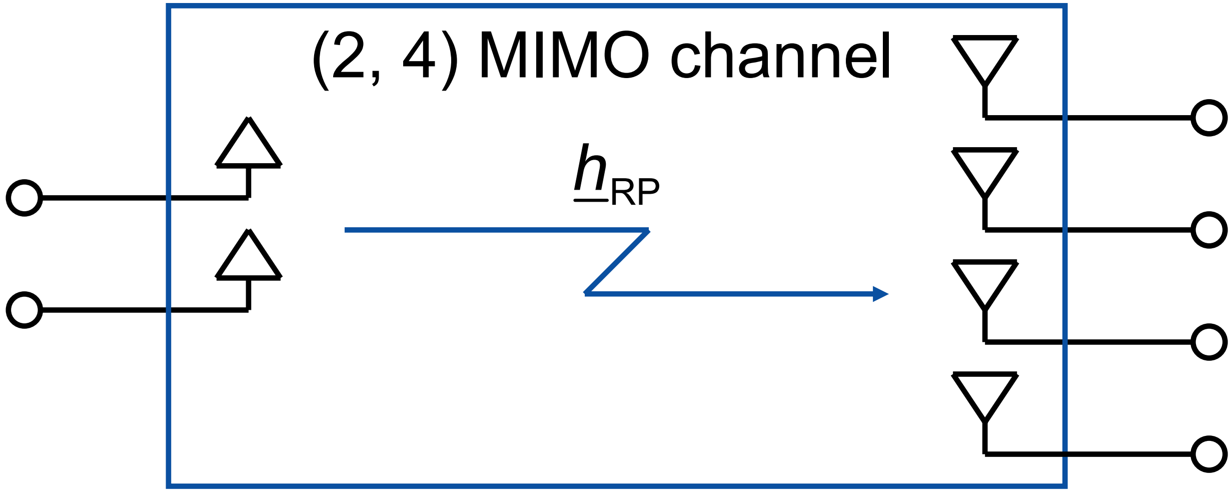
- without transmitter side channel state information

$$C = \sum_{r=1}^{R} \mathrm{ld}\left(1 + \frac{\lambda_r S}{N\sigma^2}\right) = \mathrm{ld}\left(1 + M\frac{\left|\underline{h}_\mathrm{RP}\right|^2 S}{\sigma^2}\right)$$

$\Rightarrow$ SNR gain due to receiver side beam forming,
no gain due to increased number of transmitter antennas

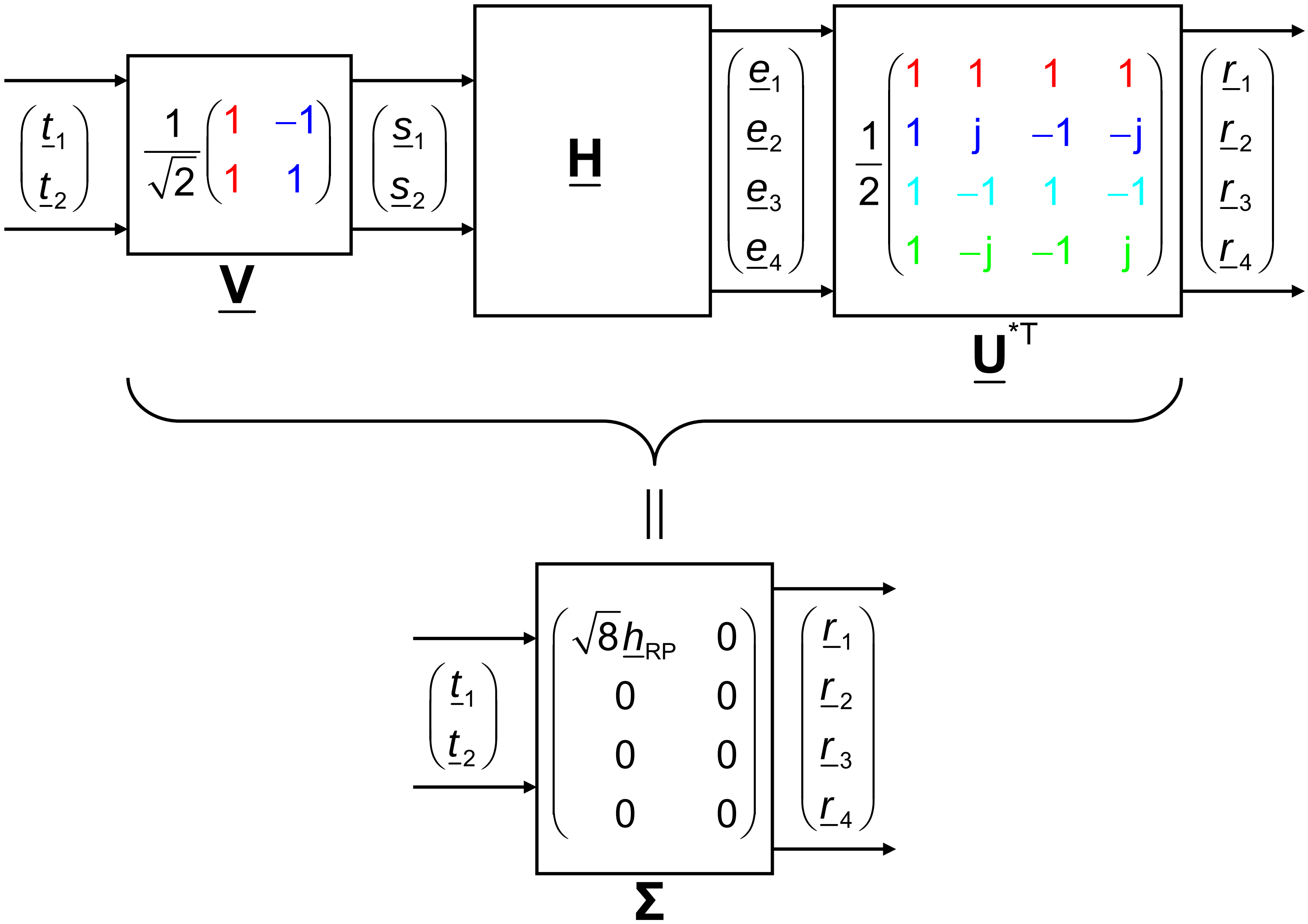
- double number of antennas $\Rightarrow$ double SNR
$\Rightarrow$ capacity gain of 1 Bit (at large SNRs)

Example: (2, 4) MIMO Channel



$$\underline{\mathbf{H}} = \underbrace{\begin{pmatrix}1\\1\\1\\1\end{pmatrix}}_{\underline{\mathbf{a}}_{Rx}} \cdot \underbrace{\left(\underline{h}_{RP}\right)}_{\underline{\mathbf{H}}_{RP}} \cdot \underbrace{\begin{pmatrix}1 & 1\end{pmatrix}}_{\underline{\mathbf{a}}_{Tx}^{T}} = \underbrace{\frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -j & -1 & j\\ 1 & -1 & 1 & -1\\ 1 & j & -1 & -j\end{pmatrix}}_{\underline{\mathbf{U}}} \cdot \underbrace{\begin{pmatrix}\sqrt{8}\,\underline{h}_{RP} & 0\\ 0 & 0\\ 0 & 0\\ 0 & 0\end{pmatrix}}_{\underline{\mathbf{\Sigma}}} \cdot \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}}_{\underline{\mathbf{V}}^{*T}}$$

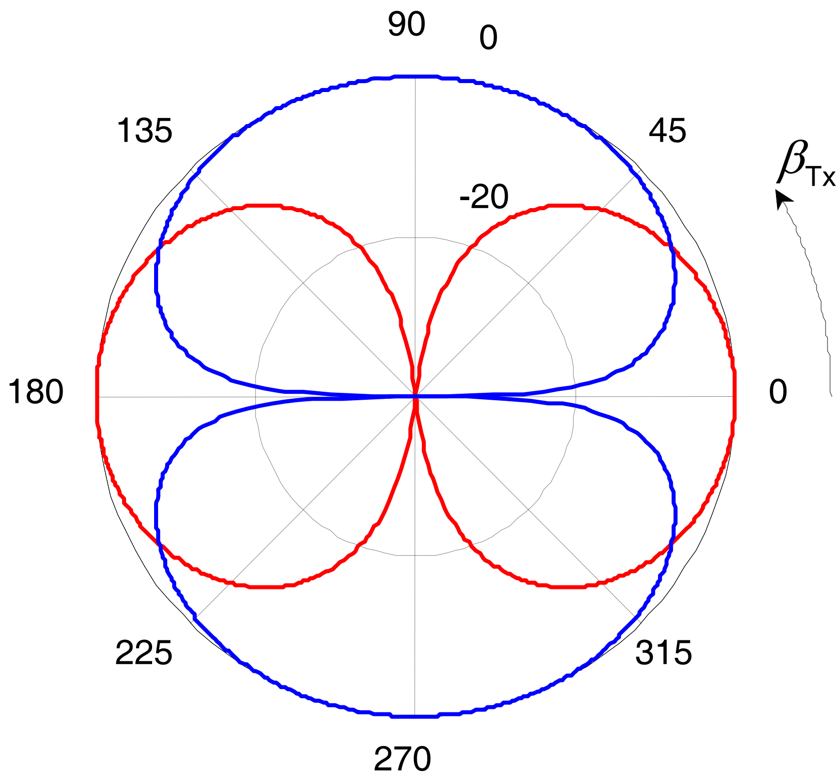
Example, System Architecture



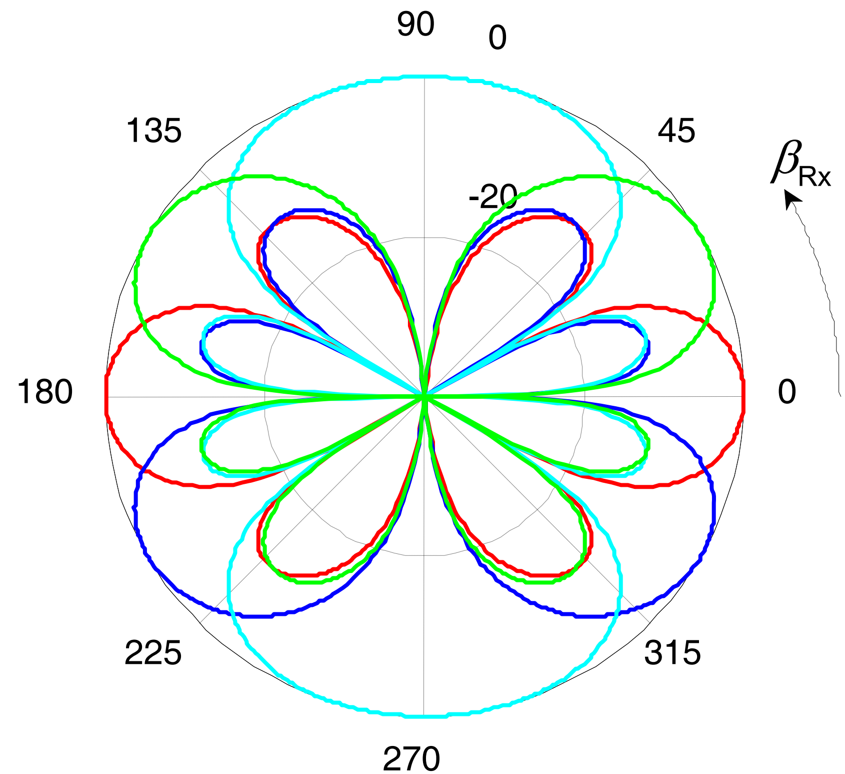
Example: Antenna Diagrams

$$10 \cdot \log\left(\frac{g_{Tx}(\beta_{Tx})}{\max\left(g_{Tx}(\beta_{Tx})\right)}\right) \Big/ \mathrm{dB}$$

$$10 \cdot \log\left(\frac{g_{Rx}(\beta_{Rx})}{\max_{Rx}\left(g(\beta_{Rx})\right)}\right) \Big/ \mathrm{dB}$$



transmitter
antenna array



receiver
antenna array

Multiple Path Channel Model

superpose channels of individual paths

$$\underline{\mathbf{H}} = \sum_{p=1}^{P} \underline{\mathbf{a}}_{\mathrm{Rx}}^{(p)} \cdot \underline{h}_{\mathrm{RP}}^{(p)} \cdot \underline{\mathbf{a}}_{\mathrm{Tx}}^{(p)\mathrm{T}} = \underbrace{\begin{pmatrix} \underline{\mathbf{a}}_{\mathrm{Rx}}^{(1)} & \cdots & \underline{\mathbf{a}}_{\mathrm{Rx}}^{(P)} \end{pmatrix}}_{\underline{\mathbf{A}}_{\mathrm{Rx}}} \cdot \underbrace{\begin{pmatrix} \underline{h}_{\mathrm{RP}}^{(1)} & & 0 \\ & \ddots & \\ 0 & & \underline{h}_{\mathrm{RP}}^{(P)} \end{pmatrix}}_{\underline{\mathbf{H}}_{\mathrm{RP}}} \cdot \underbrace{\begin{pmatrix} \underline{\mathbf{a}}_{Tx}^{(1)\mathrm{T}} \\ \vdots \\ \underline{\mathbf{a}}_{Tx}^{(P)\mathrm{T}} \end{pmatrix}}_{\underline{\mathbf{A}}_{\mathrm{Tx}}^{\mathrm{T}}}$$

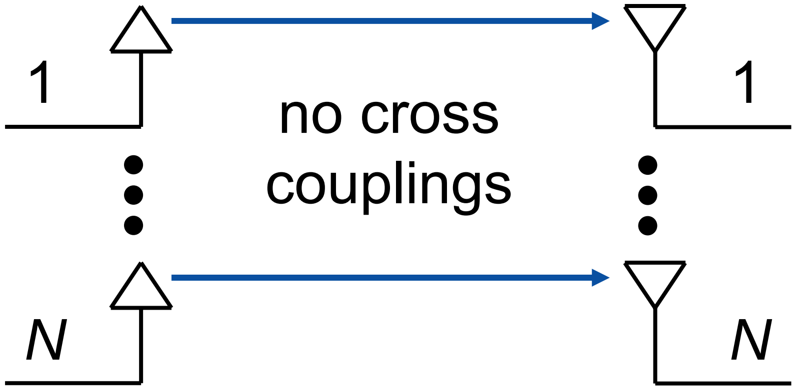
$\underline{\mathbf{A}}_{\mathrm{Rx}}$ receiver side steering matrix
$\underline{\mathbf{A}}_{\mathrm{Tx}}$ transmitter side steering matrix

$\mathrm{rank}\left(\underline{\mathbf{H}}\right) \leq \min\left(N, M, P\right)$

rich scattering: $P \to \infty$, rank not limited by number of paths

$\Rightarrow$ both beam forming and multiplexing gains possible

Flat Ribbon Cable Channel Model



$$\underline{\mathbf{H}} = \mathbf{I} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$$\operatorname{rank}(\underline{\mathbf{H}}) = N$$

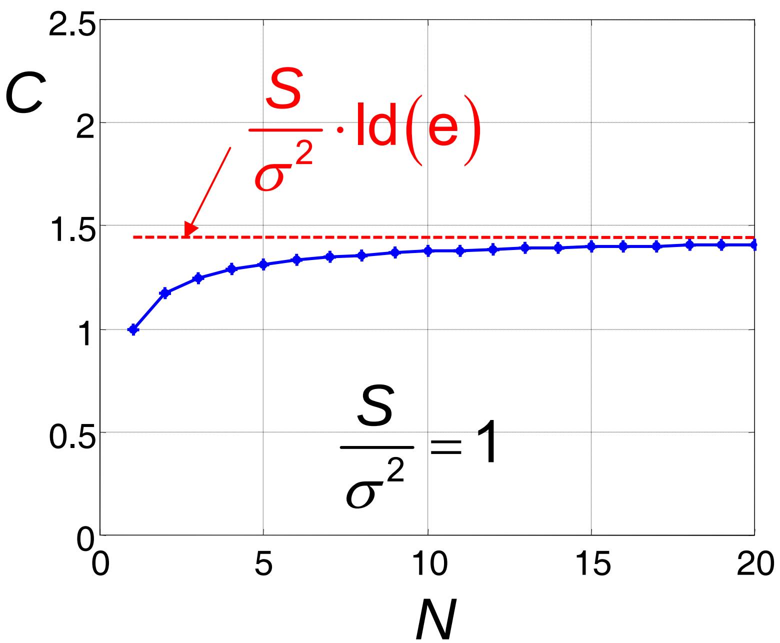
singular values: $\sqrt{\lambda_q} = 1$

all subchannels identical

$\Rightarrow$ equal power allocation is optimal

$\Rightarrow$ channel capacity with and without TxCSI is the same

$$C = N \cdot \mathrm{ld}\left(1 + \frac{S}{N\sigma^2}\right)$$



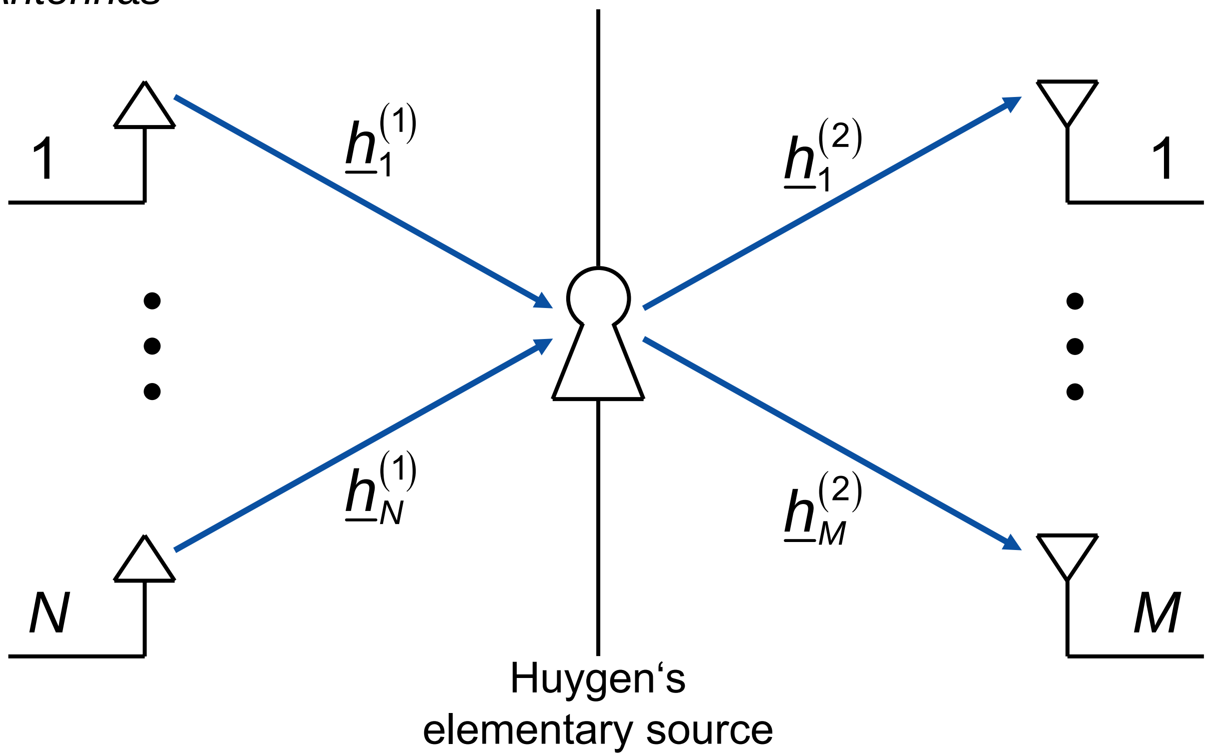
limiting value $N \to \infty$:

$$C_\infty = \lim_{N\to\infty}\left\{\frac{\mathrm{ld}\left(1+\frac{S}{N\sigma^2}\right)}{\frac{1}{N}}\right\} = \frac{S}{\sigma^2}\cdot \mathrm{ld}(e)$$

$\Rightarrow$ only (limited) gains by spatial multiplexing

Keyhole Channel Model

Chizhik (2002): *Keyholes, Correlations, and Capacities of Multielement Transmit and Receive Antennas*



$$\underline{\mathbf{h}}^{(1)} = \begin{pmatrix} \underline{h}_1^{(1)} \\ \vdots \\ \underline{h}_N^{(1)} \end{pmatrix}, \quad \underline{\mathbf{h}}^{(2)} = \begin{pmatrix} \underline{h}_1^{(2)} \\ \vdots \\ \underline{h}_M^{(2)} \end{pmatrix} \quad \Rightarrow \quad \underline{\mathbf{H}} = \begin{pmatrix} \underline{h}_1^{(2)}\underline{h}_1^{(1)} & \cdots & \underline{h}_1^{(2)}\underline{h}_N^{(1)} \\ \vdots & & \vdots \\ \underline{h}_M^{(2)}\underline{h}_1^{(1)} & \cdots & \underline{h}_M^{(2)}\underline{h}_N^{(1)} \end{pmatrix} = \underline{\mathbf{h}}^{(2)} \cdot \underline{\mathbf{h}}^{(1)\mathrm{T}}$$

Singular Value Decomposition of the Keyhole Channel

$$\underline{\mathbf{H}} = \underbrace{\begin{pmatrix} \dfrac{\underline{\mathbf{h}}^{(2)}}{\left\|\underline{\mathbf{h}}^{(2)}\right\|} & \underbrace{\cdots\cdots\cdots}_{N\text{-1 orthonormal columns}} \end{pmatrix}}_{\underline{\mathbf{U}}} \cdot \underbrace{\begin{pmatrix} \left\|\underline{\mathbf{h}}^{(1)}\right\|\left\|\underline{\mathbf{h}}^{(2)}\right\| & 0 & \cdots & 0 \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 0 \end{pmatrix}}_{\mathbf{\Sigma}} \cdot \underbrace{\begin{pmatrix} \dfrac{\underline{\mathbf{h}}^{(1)\mathrm{T}}}{\left\|\underline{\mathbf{h}}^{(1)}\right\|} \\ \vdots\Big\} \; M-1 \text{ orthonormal rows} \end{pmatrix}}_{\underline{\mathbf{V}}^{*\mathrm{T}}}$$

$$\Rightarrow \sqrt{\lambda_1} = \left\|\underline{\mathbf{h}}^{(1)}\right\|\left\|\underline{\mathbf{h}}^{(2)}\right\|, \quad \sqrt{\lambda_2} = \ldots = \sqrt{\lambda_Q} = 0$$

$\Rightarrow \mathrm{rank}\left(\underline{\mathbf{H}}\right) = 1$ rank deficient!

$\Rightarrow$ optimum signal processing strategy consists in transmitter and receiver side matched filtering

Capacity of the Keyhole Channel

- with transmitter side channel state information

$$C = \mathrm{ld}\left(1 + \frac{S \cdot \left\|\underline{\mathbf{h}}^{(1)}\right\|^2 \cdot \left\|\underline{\mathbf{h}}^{(2)}\right\|^2}{\sigma^2}\right)$$

$\Rightarrow$ both transmitter and receiver side beam forming gains

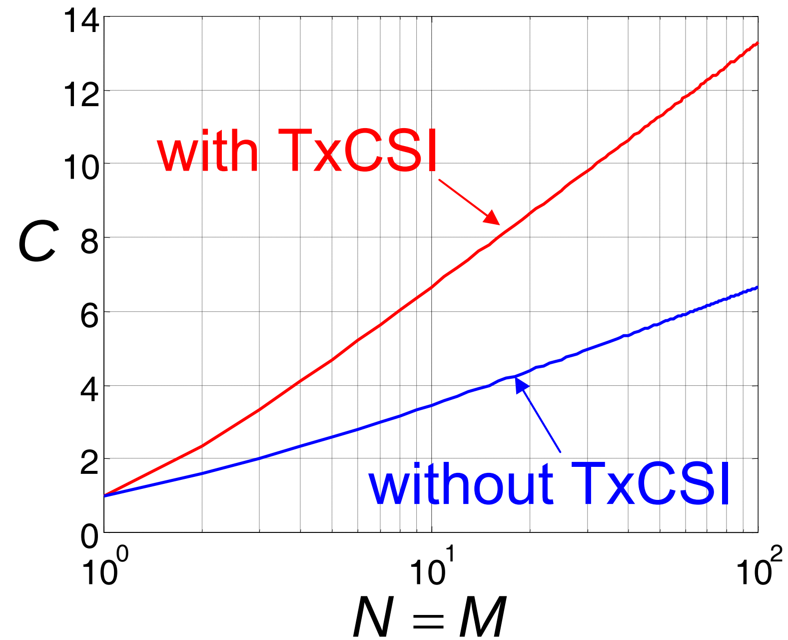
- without transmitter side channel state information

$$C = \mathrm{ld}\left(1 + \frac{S \cdot \left\|\underline{\mathbf{h}}^{(1)}\right\|^2 \cdot \left\|\underline{\mathbf{h}}^{(2)}\right\|^2}{N \cdot \sigma^2}\right)$$

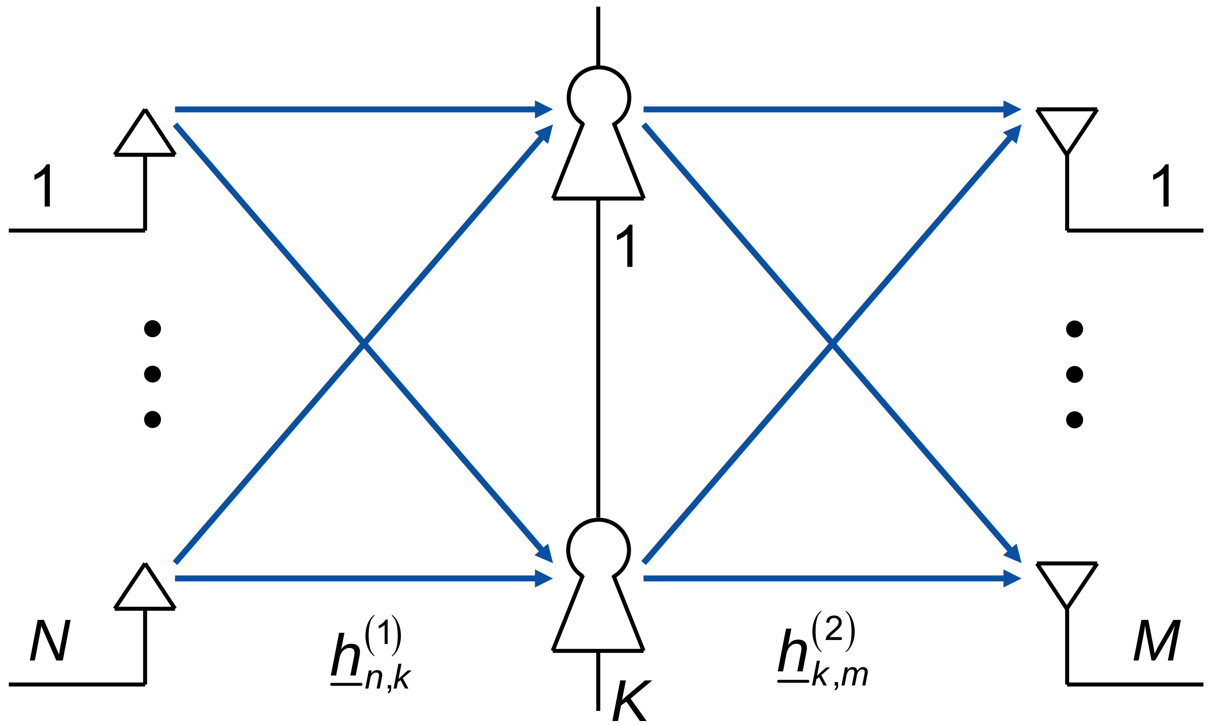
$\Rightarrow$ only receiver side beam forming gains

$$\underline{h}_n^{(1)} = 1, \quad \underline{h}_m^{(2)} = 1$$

$$\frac{S}{\sigma^2} = 1$$



Channel Model with Multiple Keyholes



$$\underline{\mathbf{h}}^{(1,k)} = \begin{pmatrix} \underline{h}_{1,k}^{(1)} \\ \vdots \\ \underline{h}_{N,k}^{(1)} \end{pmatrix}, \quad \underline{\mathbf{h}}^{(2,k)} = \begin{pmatrix} \underline{h}_{k,1}^{(2)} \\ \vdots \\ \underline{h}_{k,M}^{(2)} \end{pmatrix} \quad \Rightarrow \quad \underline{\mathbf{H}}^{(k)} = \underline{\mathbf{h}}^{(2,k)} \cdot \underline{\mathbf{h}}^{(1,k)\mathrm{T}} \Rightarrow \underline{\mathbf{H}} = \sum_{k=1}^{K} \underline{\mathbf{H}}^{(k)}$$

$$\mathrm{rank}\left(\underline{\mathbf{H}}\right) \leq \min\left(N, M, K\right)$$

Channel Model with Independent Fading

$$\underline{\mathbf{H}} = \begin{pmatrix} \underline{h}_{1,1} & \cdots & \underline{h}_{1,N} \\ \vdots & & \vdots \\ \underline{h}_{M,1} & \cdots & \underline{h}_{M,N} \end{pmatrix}$$

channel coefficients independent identically Gaussian distributed

$$\underline{h}_{m,n} \sim \mathbb{C}\mathcal{N}\left\{0, \sigma_{\mathrm{h}}^2\right\}, \quad \text{here: } \underline{h}_{m,n} \sim \mathbb{C}\mathcal{N}\left\{0, 1\right\}$$

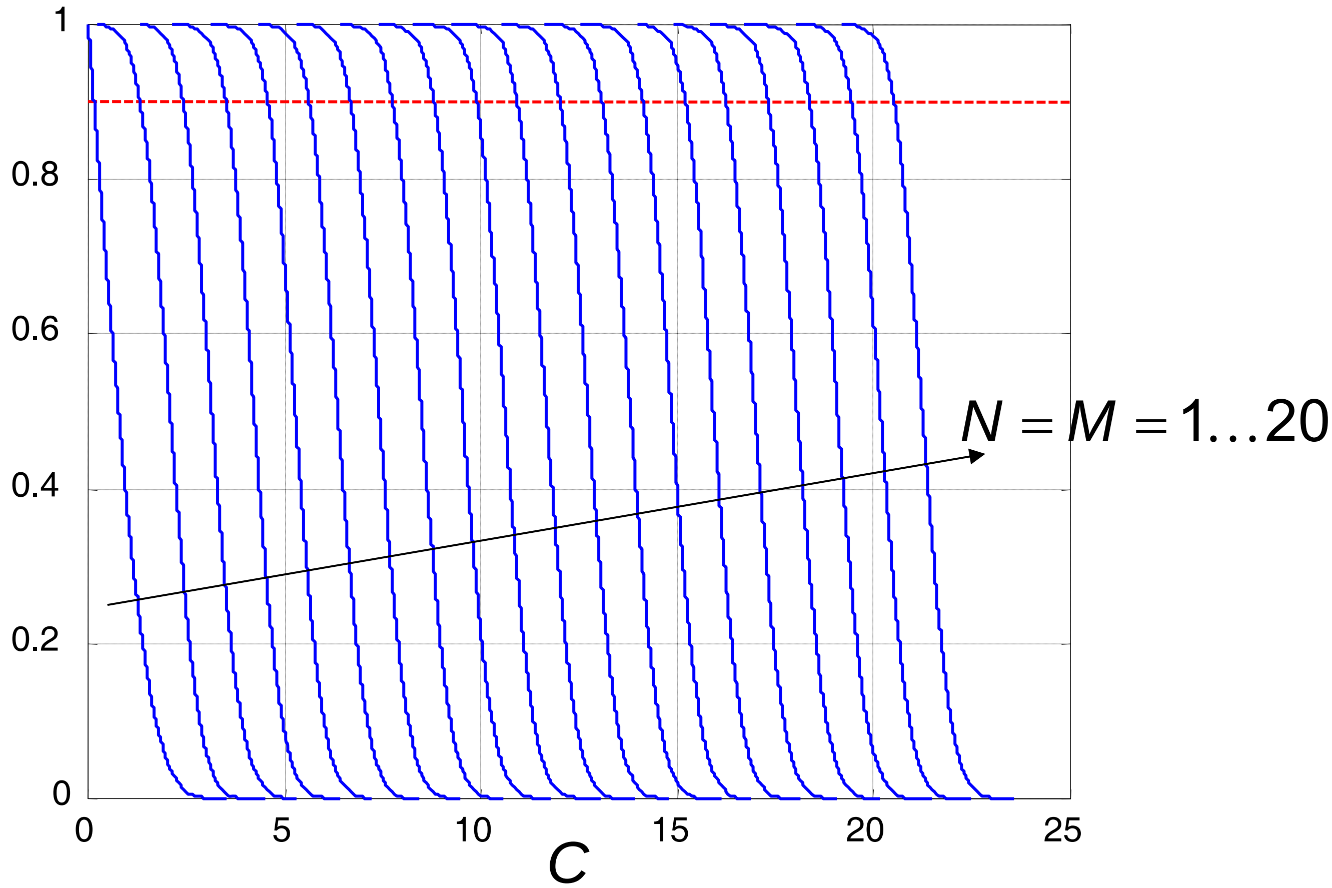
Channel capacity is a function of the eigenvalues of the Wishart-Matrix:

$$\underline{\mathbf{W}} = \begin{cases} \underline{\mathbf{H}}\underline{\mathbf{H}}^{*\mathrm{T}} & M > N \\ \underline{\mathbf{H}}^{*\mathrm{T}}\underline{\mathbf{H}} & M \le N \end{cases}$$

$\Rightarrow$ random matrix theory, see : Metha: *Random Matrices*
eigenvalues are Wishart distributed

Instantaneous Channel Capacity with Transmitter Side Channel State Information

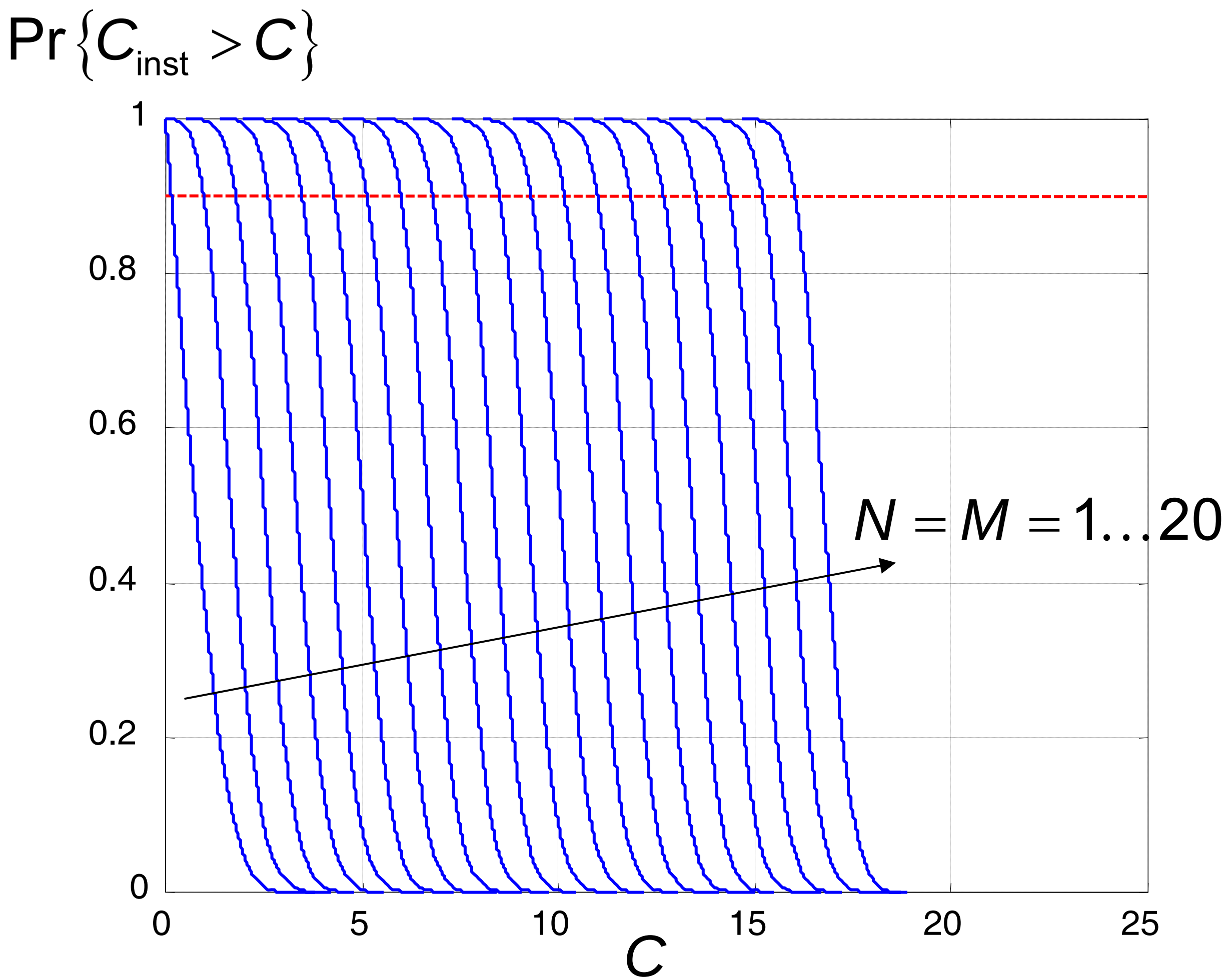
$\Pr\{C_{\text{inst}} > C\}$



$$\frac{S}{\sigma^2} = 1$$

$$\sigma_h^2 = 1$$

Instantaneous Channel Capacity without Transmitter Side Channel State Information

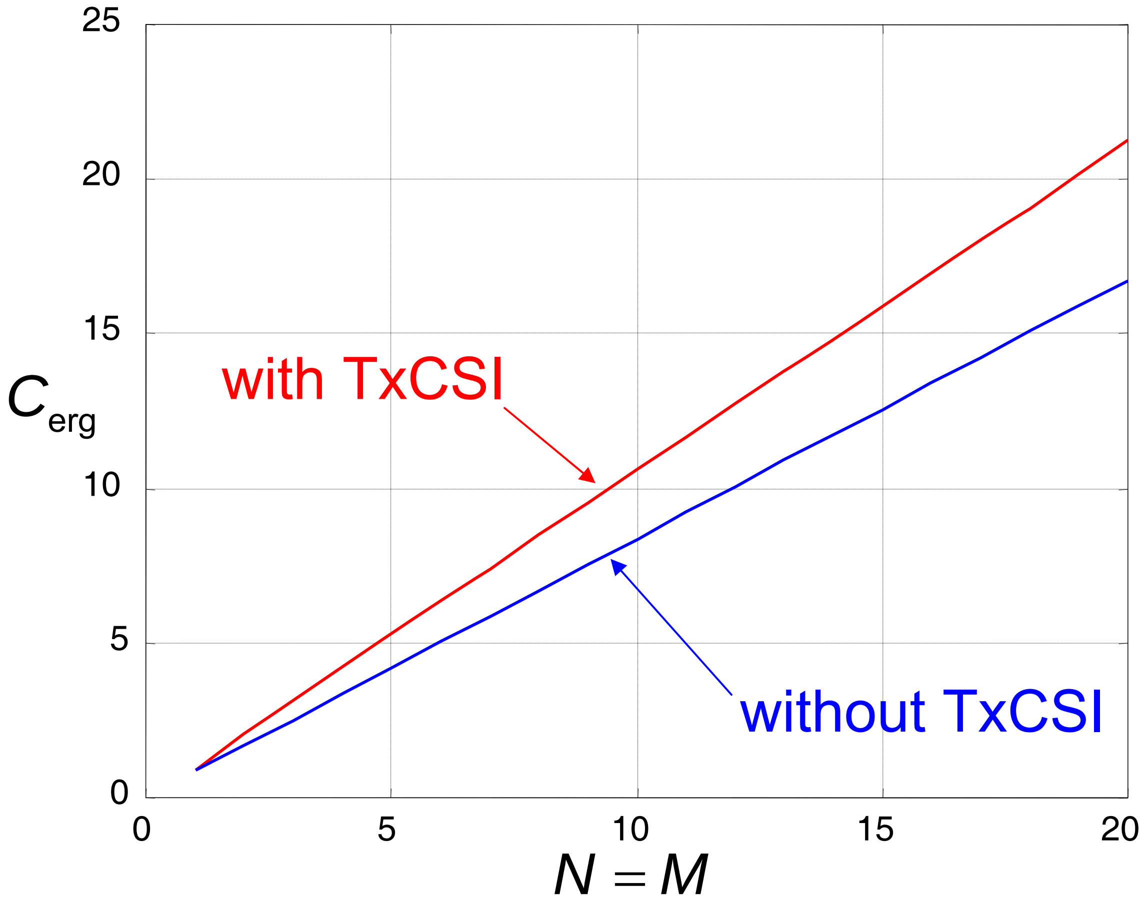


$$\frac{S}{\sigma^2} = 1$$

$$\sigma_h^2 = 1$$

Foschini, Gans (1998): *On Limits of Wireless Communications in a Fading Environment when Using Multiple Antennas*

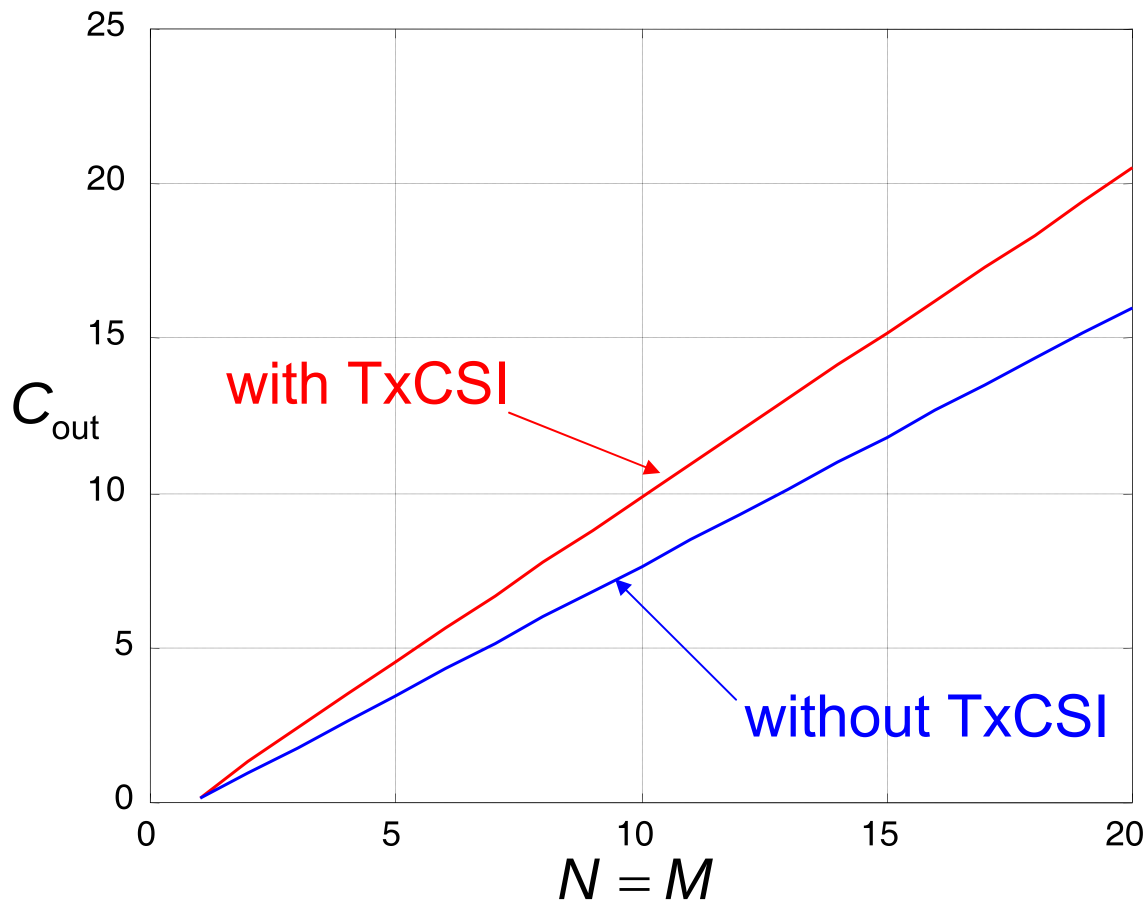
Ergodic Channel Capacity



$$\frac{S}{\sigma^2} = 1$$

$$\sigma_h^2 = 1$$

Outage Channel Capacity

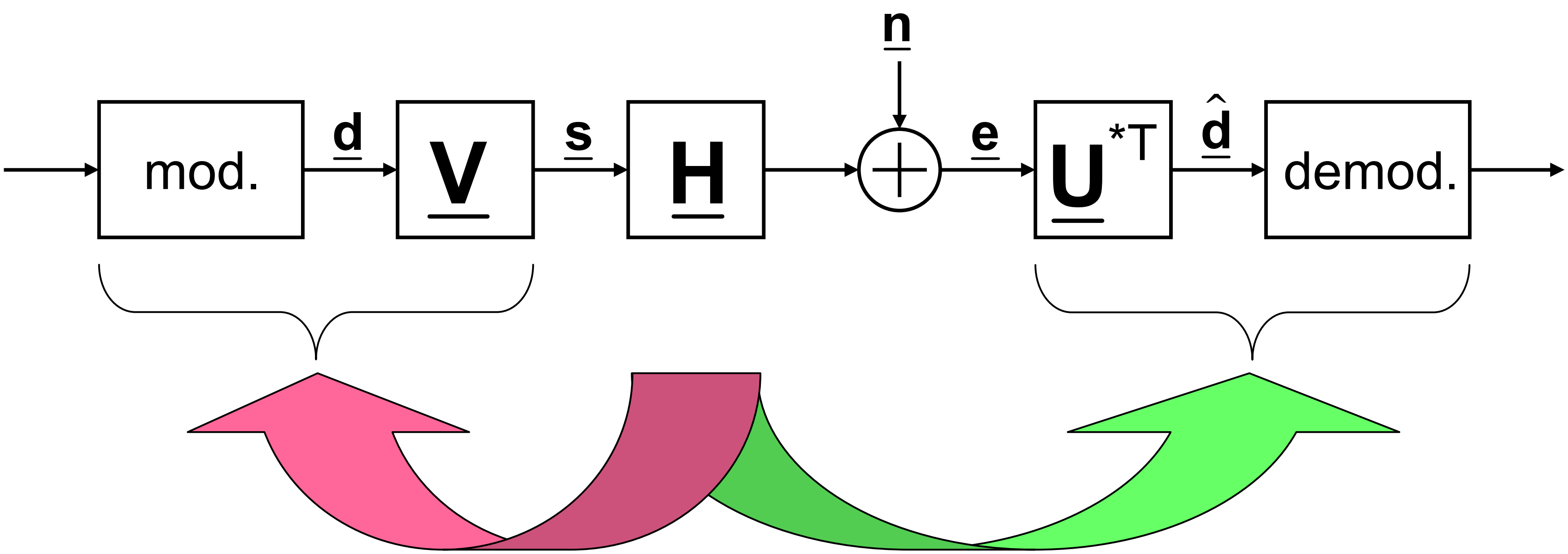


$$P_{out} = 0{,}1$$

$$\frac{S}{\sigma^2} = 1$$

$$\sigma_h^2 = 1$$

Block Diagram



transmitter side channel state information

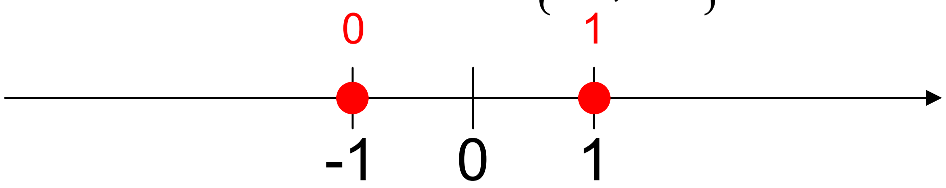
- e.g. by signaling back CSI or exploiting channel reciprocity in TDD systems
- critical

receiver side channel state information

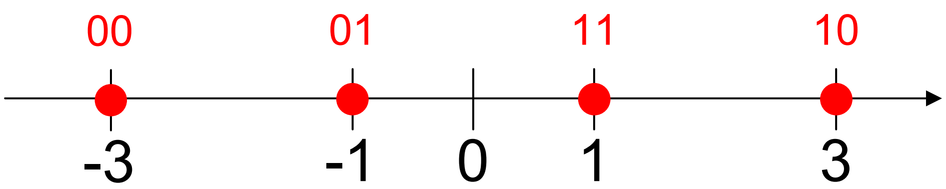
- e.g. by training signal based channel estimation
- uncritical

Pulse Amplitude Modulation (PAM)

- 2-PAM = BPSK $\mathbb{D} = \{-1, +1\}$



- 4-PAM $\mathbb{D} = \{-3, -1, +1, +3\}$



average symbol energy:

$$E_S = \frac{1}{M}\sum_{m=1}^{M}(2m-1-M)^2$$

$$= \frac{1}{3}\left(M^2-1\right)$$

average SNR of real part:

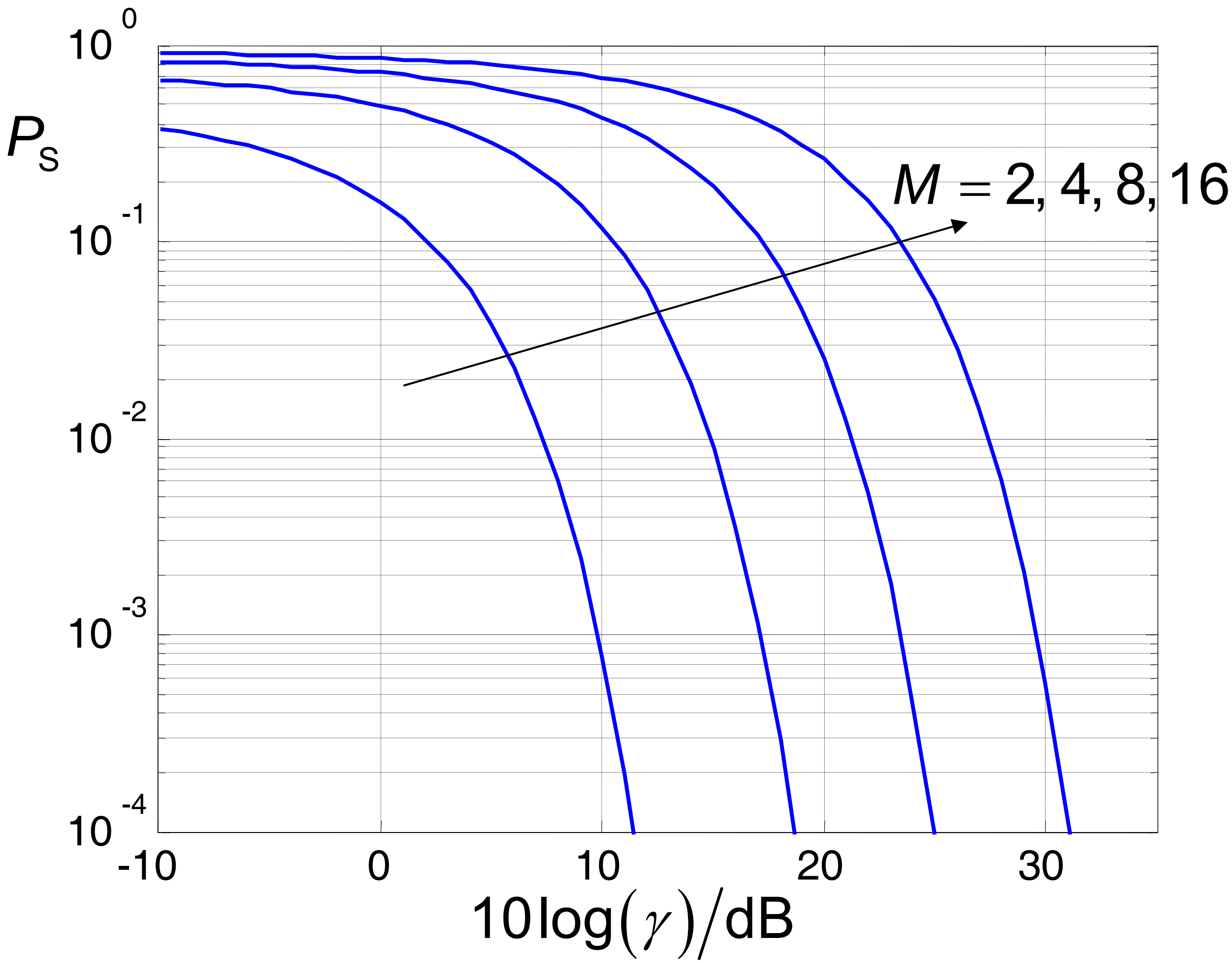
$$\gamma = \frac{2E_S}{\sigma^2}$$

in general: M-ary Pam, $M = 2^B$

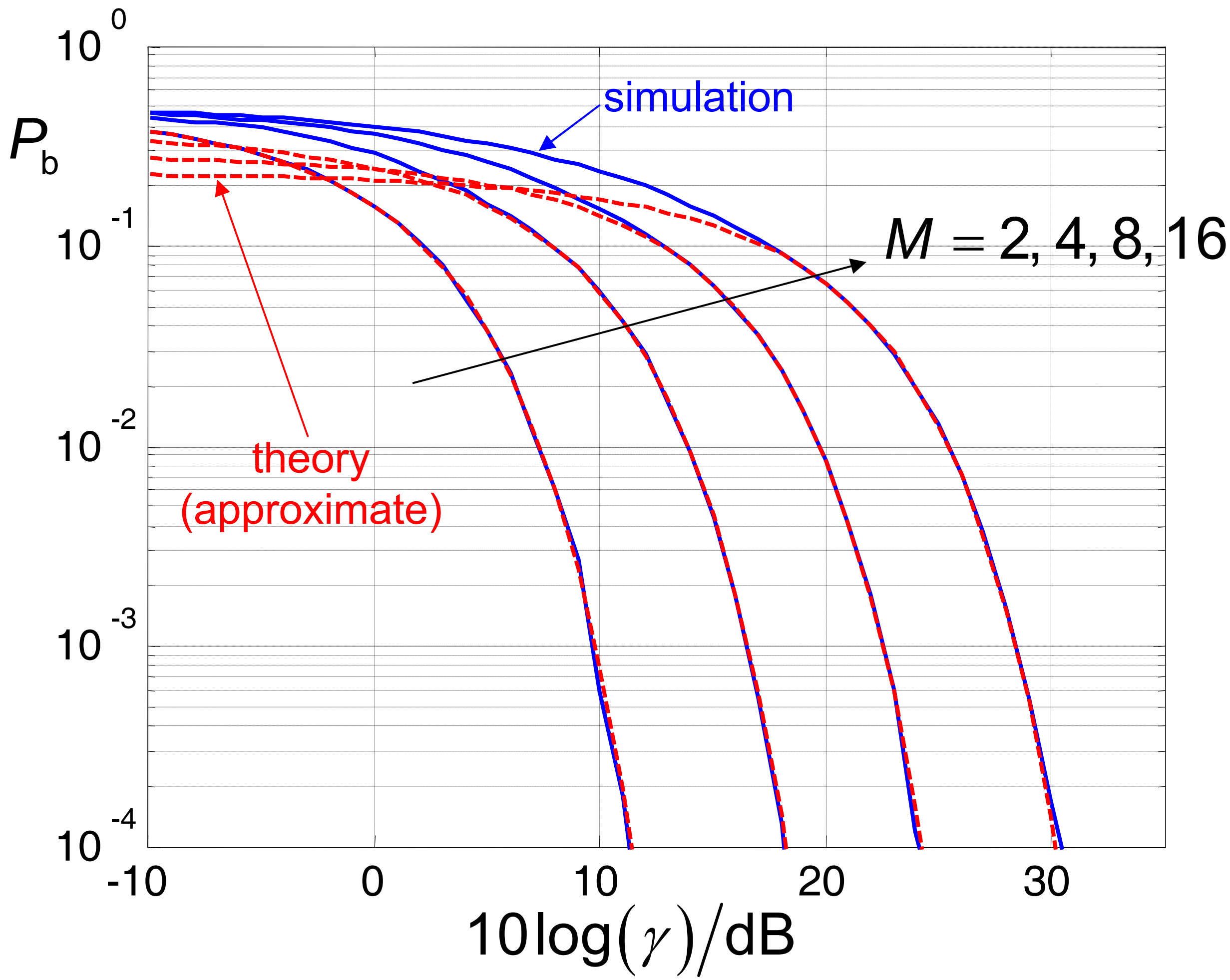
symbol error probability: $P_S = \frac{M-1}{M}\cdot \mathrm{erfc}\left(\sqrt{\frac{3\gamma}{2\left(M^2-1\right)}}\right)$

bit error probability (Gray coded): $P_b \approx \frac{1}{B}P_S$

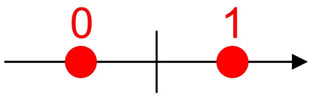
Symbol Error Performance of PAM



Bit Error Performance of PAM

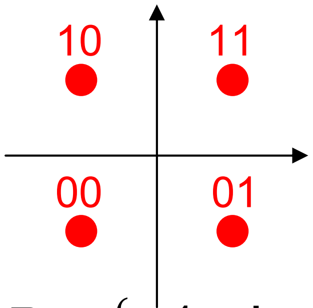


- 2-RQAM = 2-PAM = BPSK



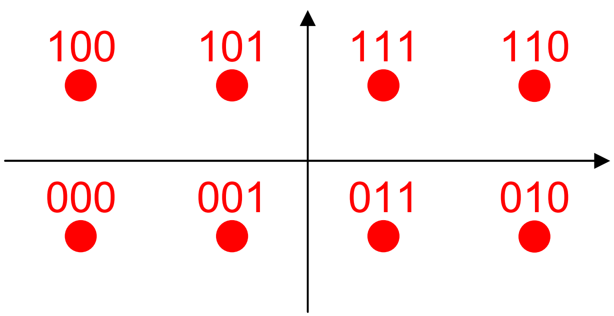
$$\mathbb{D} = \{-1, +1\}$$

- 4-RQAM = 4-QAM = QPSK

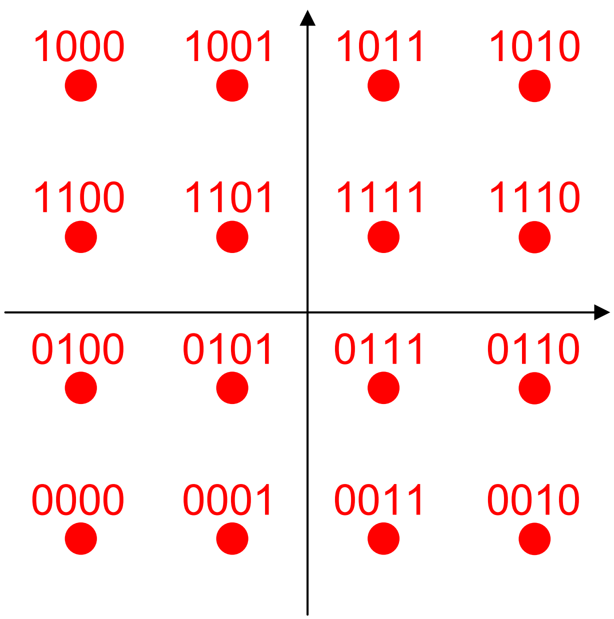


$$\mathbb{D} = \{-1-j, -1+j, +1-j, +1+j\}$$

- 8-RQAM



- 16-RQAM = 16-QAM



Analysis of RQAM

RQAM consists of a M_R-ary PAM for the real part and a M_I-ary PAM for the imaginary part, $M_\mathrm{R} \cdot M_\mathrm{I} = 2^B$

⇒ average symbol energy is the sum of the symbol energies in real and imaginary part

$$E_\mathrm{S} = \frac{1}{3}\left(M_\mathrm{R}^2 - 1\right) + \frac{1}{3}\left(M_\mathrm{I}^2 - 1\right) = \frac{1}{3}\left(M_\mathrm{R}^2 + M_\mathrm{I}^2 - 2\right), \quad \gamma = \frac{E_\mathrm{S}}{\sigma^2}$$

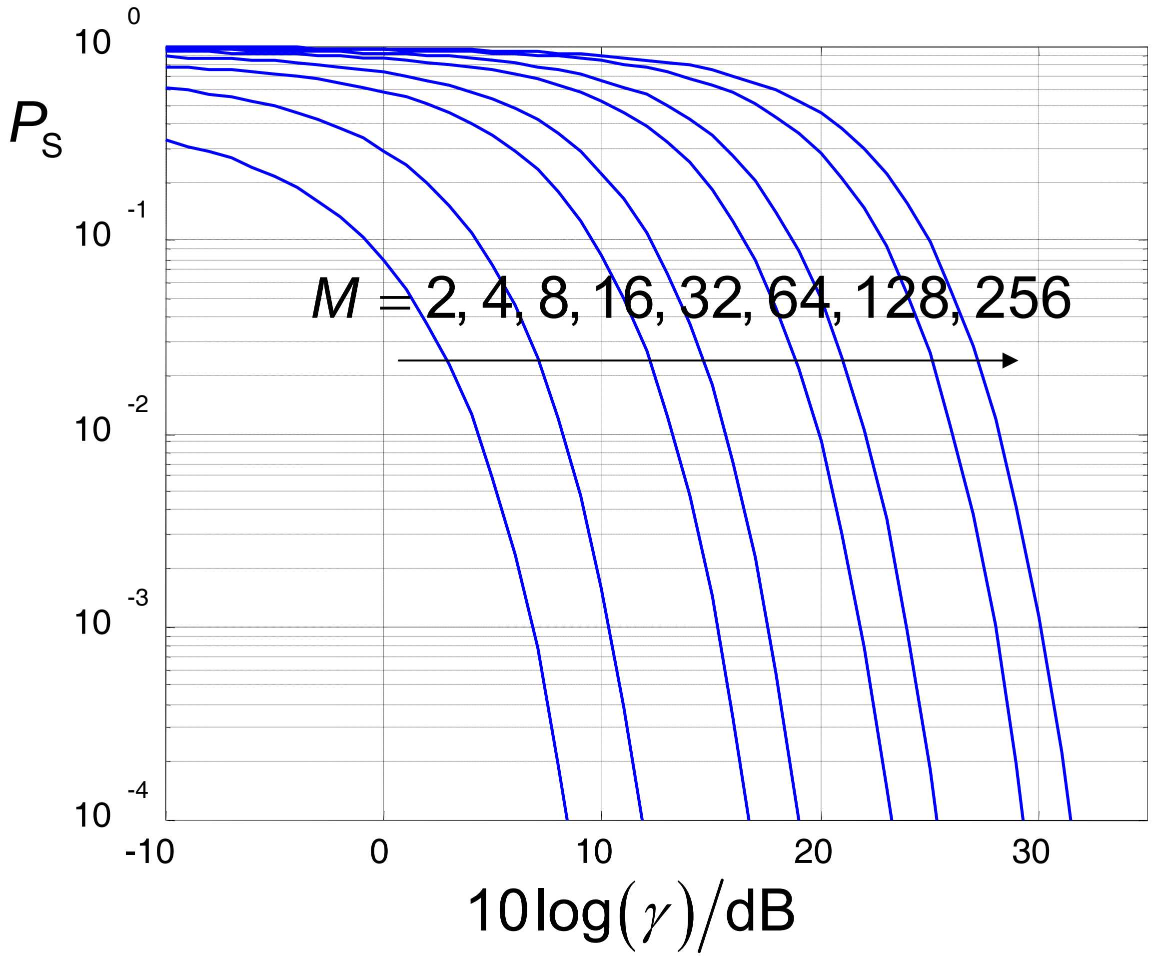
⇒ no symbol error occurs if there is neither a symbol error in the real part nor in the imaginary part

$$1 - P_\mathrm{S} = \left(1 - P_\mathrm{S,R}\right)\left(1 - P_\mathrm{S,I}\right)$$

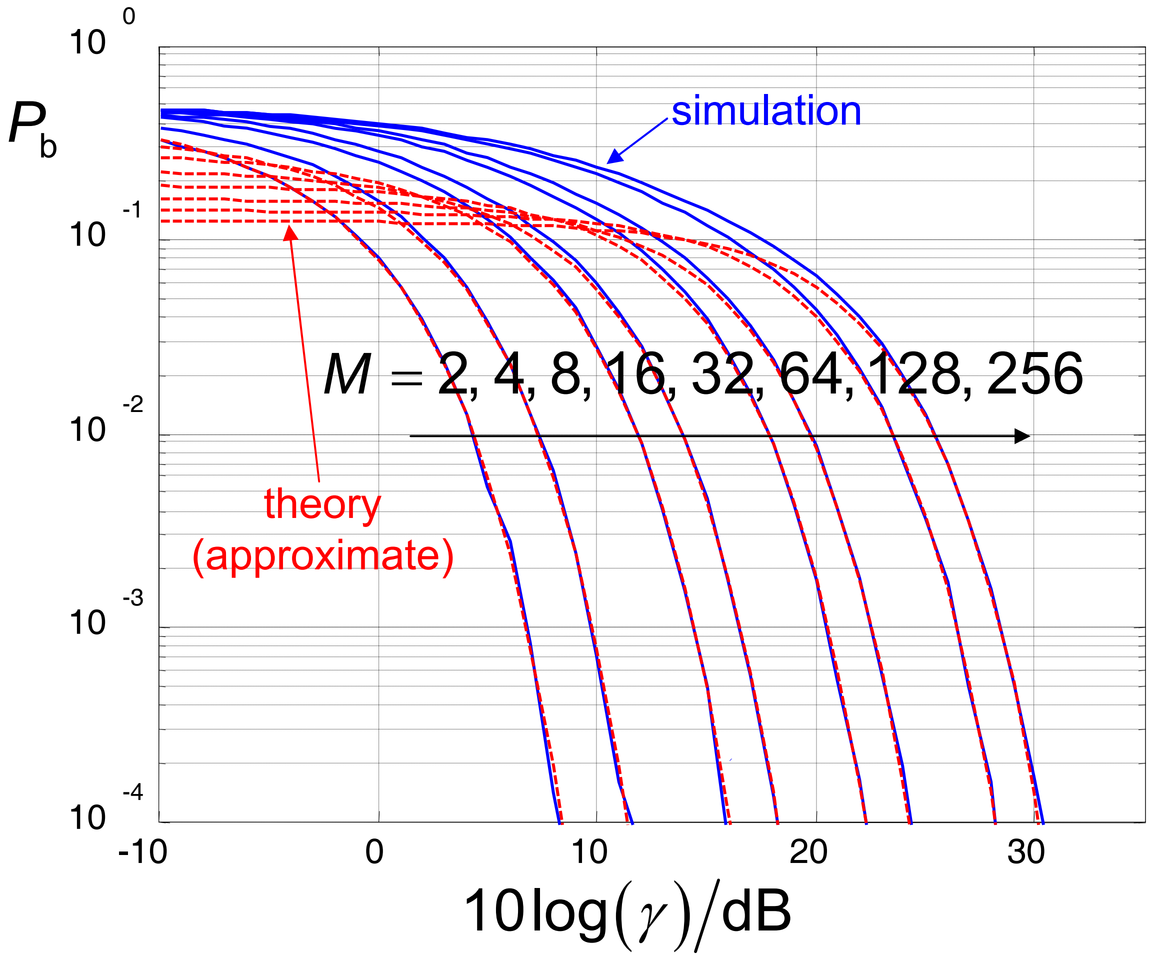
$$P_\mathrm{S} = 1 - \left[1 - \frac{M_\mathrm{R} - 1}{M_\mathrm{R}} \operatorname{erfc}\left(\sqrt{\frac{3\gamma}{M_\mathrm{R}^2 + M_\mathrm{I}^2 - 2}}\right)\right]\left[1 - \frac{M_\mathrm{I} - 1}{M_\mathrm{I}} \operatorname{erfc}\left(\sqrt{\frac{3\gamma}{M_\mathrm{R}^2 + M_\mathrm{I}^2 - 2}}\right)\right]$$

$$P_\mathrm{b} \approx \frac{1}{B} P_\mathrm{S}$$

Symbol Error Performance of RQAM



Bit Error Performance of RQAM



Adaptive Modulation

For given maximum acceptable bit error probability P_{bmax} the number of bits which can be transmitted depends on the SNR γ!

e.g. $P_{\text{bmax}} = 10^{-3}$

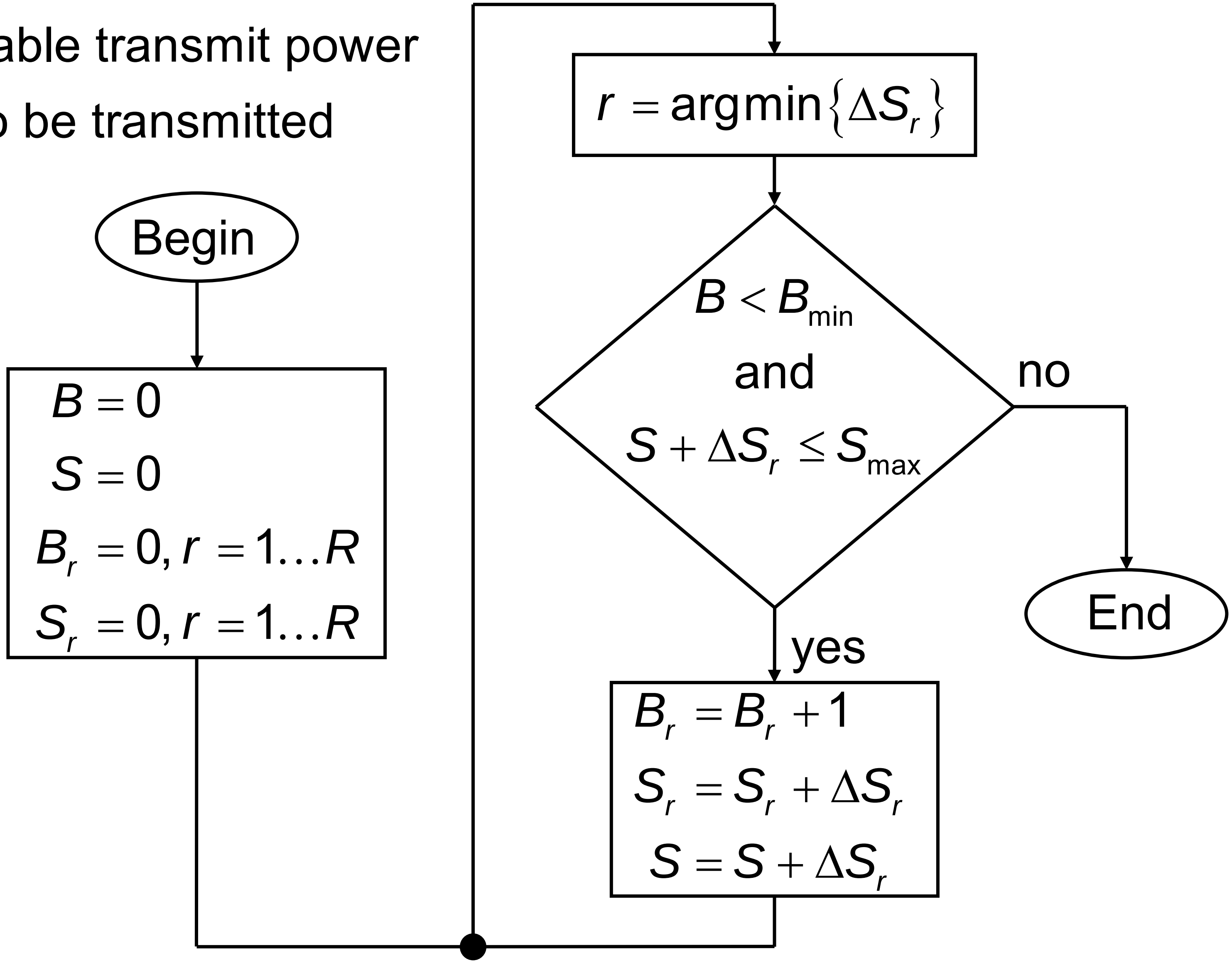
B	1	2	3	4	5	6	7	8	9	10
$M = 2^B$	2	4	8	16	32	64	128	256	512	1024
$10\log(\gamma)/\text{dB}$	6,8	9,8	14	17	21	23	26	28	32	34

Transmit power increments $\Delta S_r = \Delta\gamma \cdot \sigma^2 / \lambda_r$ required for transmitting one additional bit depend on the channel quality and the number B of already transmitted bits!

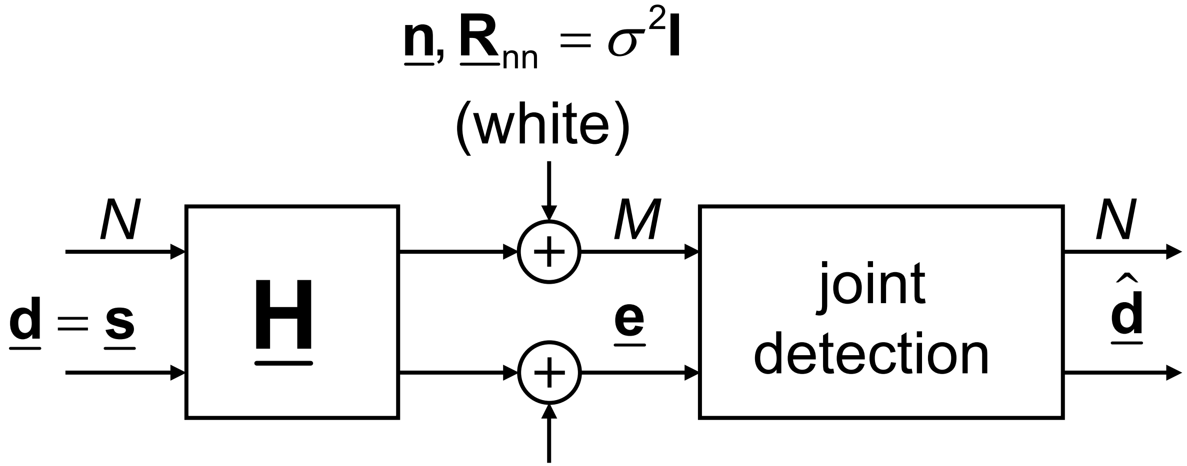
Hughes Hartogs Algorithmus

S_{max}: available transmit power

B_{min}: bits to be transmitted



System Model



$$\underline{\mathbf{e}} = \underline{\mathbf{H}} \cdot \underline{\mathbf{s}} + \underline{\mathbf{n}} = \underline{\mathbf{H}} \cdot \underline{\mathbf{d}} + \underline{\mathbf{n}}$$

for good performance: $M \geq N$

no transmitter side cooperation,
only receiver side cooperation

exploit discrete nature of the modulation alphabet

- maximum a posteriori criterion (MAP):

$$\hat{\underline{\mathbf{d}}} = \underset{\underline{\mathbf{d}} \in \mathbb{D}^N}{\operatorname{argmax}} \left\{ \Pr\left\{ \underline{\mathbf{d}} \middle| \underline{\mathbf{e}} \right\} \right\} = \underset{\underline{\mathbf{d}} \in \mathbb{D}^N}{\operatorname{argmax}} \left\{ p\left(\underline{\mathbf{e}} \middle| \underline{\mathbf{d}} \right) \cdot \Pr\left\{ \underline{\mathbf{d}} \right\} \right\}$$

- maximum likelihood criterion (ML):

$$\hat{\underline{\mathbf{d}}} = \underset{\underline{\mathbf{d}} \in \mathbb{D}^N}{\operatorname{argmax}} \left\{ p\left(\underline{\mathbf{e}} \middle| \underline{\mathbf{d}} \right) \right\}$$

$$= \underset{\underline{\mathbf{d}} \in \mathbb{D}^N}{\operatorname{argmin}} \left\{ \left\| \underline{\mathbf{e}} - \underline{\mathbf{H}} \cdot \underline{\mathbf{d}} \right\|^2 \right\} \quad \text{for Gaussian noise } \underline{\mathbf{n}}$$

Receive Zero Forcing (ZF) (1)

do not restrict search to discrete elements of the modulation alphabet

$$\hat{\underline{\mathbf{d}}} = \underset{\underline{\mathbf{d}} \in \mathbb{C}^N}{\arg\min} \left\{ \left\| \underline{\mathbf{e}} - \underline{\mathbf{H}} \cdot \underline{\mathbf{d}} \right\|^2 \right\}$$

$$= \underset{\underline{\mathbf{d}} \in \mathbb{C}^N}{\arg\min} \left\{ \underline{\mathbf{e}}^{*\mathrm{T}} \underline{\mathbf{e}} - \underline{\mathbf{d}}^{*\mathrm{T}} \underline{\mathbf{H}}^{*\mathrm{T}} \underline{\mathbf{e}} - \underline{\mathbf{e}}^{*\mathrm{T}} \underline{\mathbf{H}} \underline{\mathbf{d}} + \underline{\mathbf{d}}^{*\mathrm{T}} \underline{\mathbf{H}}^{*\mathrm{T}} \underline{\mathbf{H}} \underline{\mathbf{d}} \right\}$$

Definition:
Let $\underline{\mathbf{x}}$ be a complex vector with the elements $\underline{x}_n = x_{\mathrm{R},n} + \mathrm{j} \cdot x_{\mathrm{I},n}$
and f($\underline{\mathbf{x}}$) be a scalar function of $\underline{\mathbf{x}}$.
One defines the generalized derivative of f($\underline{\mathbf{x}}$) with respect to $\underline{\mathbf{x}}$ as the *N* dimensional vector

$$\nabla_{\mathrm{x}} \mathrm{f} = \begin{pmatrix} \dfrac{\mathrm{d}f}{\mathrm{d}\underline{x}_1} \\ \vdots \\ \dfrac{\mathrm{d}f}{\mathrm{d}\underline{x}_N} \end{pmatrix} \quad \text{with} \quad \frac{\mathrm{d}f}{\mathrm{d}\underline{x}_n} = \frac{1}{2}\left(\frac{\partial \mathrm{f}}{\partial x_{\mathrm{R},n}} - \mathrm{j}\frac{\partial \mathrm{f}}{\partial x_{\mathrm{I},n}} \right)$$

Rules:

$$\nabla_{\mathrm{x}}\left(\underline{c}\right) = 0 \qquad \nabla_{\mathrm{x}}\left(\underline{\mathbf{x}}^{*\mathrm{T}}\underline{\mathbf{a}}\right) = 0$$

$$\nabla_{\mathrm{x}}\left(\underline{\mathbf{a}}^{*\mathrm{T}}\underline{\mathbf{x}}\right) = \underline{\mathbf{a}}^{*} \qquad \nabla_{\mathrm{x}}\left(\underline{\mathbf{x}}^{*\mathrm{T}}\underline{\mathbf{A}}\underline{\mathbf{x}}\right) = \underline{\mathbf{A}}^{\mathrm{T}}\underline{\mathbf{x}}^{*}$$

Receive Zero Forcing (ZF) (2)

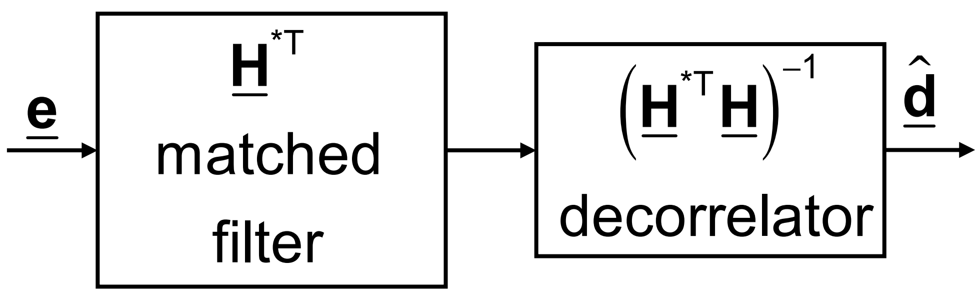
$$\nabla_d \left(\underline{\mathbf{e}}^{*T}\underline{\mathbf{e}} - \underline{\mathbf{d}}^{*T}\underline{\mathbf{H}}^{*T}\underline{\mathbf{e}} - \underline{\mathbf{e}}^{*T}\underline{\mathbf{H}}\underline{\mathbf{d}} + \underline{\mathbf{d}}^{*T}\underline{\mathbf{H}}^{*T}\underline{\mathbf{H}}\underline{\mathbf{d}} \right)$$

$$= -\underline{\mathbf{H}}^{T}\underline{\mathbf{e}}^{*} + \underline{\mathbf{H}}^{T}\underline{\mathbf{H}}^{*}\underline{\mathbf{d}}^{*} = 0$$

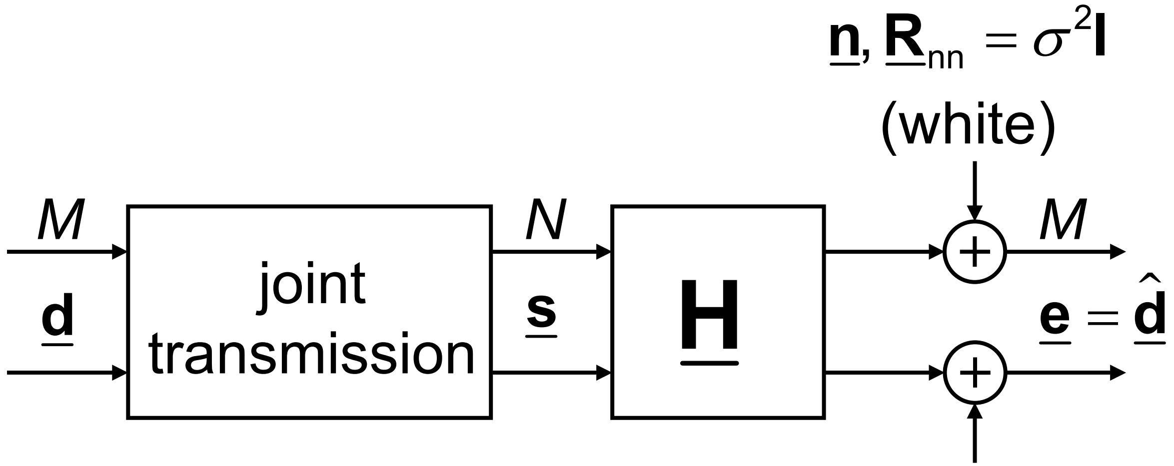
$$\Rightarrow \hat{\underline{\mathbf{d}}} = \left(\underline{\mathbf{H}}^{*T}\underline{\mathbf{H}}\right)^{-1}\underline{\mathbf{H}}^{*T} \cdot \underline{\mathbf{e}}$$

$$\underline{\mathbf{D}}_{ZF} = \left(\underline{\mathbf{H}}^{*T}\underline{\mathbf{H}}\right)^{-1} \cdot \underline{\mathbf{H}}^{*T}$$

left pseudoinverse



System Model



$$\underline{\mathbf{e}} = \underline{\hat{\mathbf{d}}} = \underline{\mathbf{H}} \cdot \underline{\mathbf{s}} + \underline{\mathbf{n}}$$

for good performance: $N \geq M$

only transmitter side cooperation,
no receiver side cooperation

Transmit Zero Forcing (ZF) (1)

Design a transmitted signal of minimum energy resulting in interference free data estimates!

minimize $\mathrm{f}\left(\underline{\mathbf{s}}\right) = S = \left\|\underline{\mathbf{s}}\right\|^2 = \underline{\mathbf{s}}^{*\mathrm{T}}\,\underline{\mathbf{s}}$

subject to the M constraints

$$\underline{\mathbf{d}} = \underline{\mathbf{H}} \cdot \underline{\mathbf{s}} \quad \Leftrightarrow \quad \mathrm{g}\left(\underline{\mathbf{s}}\right) = \underline{\mathbf{d}} - \underline{\mathbf{H}} \cdot \underline{\mathbf{s}} = 0$$

$$\mathrm{g}_m\left(\underline{\mathbf{s}}\right) = \underline{\mathbf{d}}_m - \underline{\mathbf{H}}_m \cdot \underline{\mathbf{s}} = 0$$

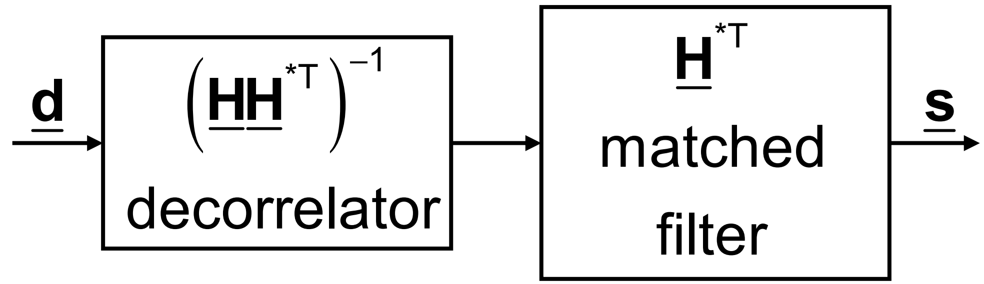
$\Rightarrow$ Lagrangian multipliers

$$\operatorname{grad} \mathrm{f}\left(\underline{\mathbf{s}}\right) + \sum_{m=1}^{M} \underline{\lambda}_m \cdot \operatorname{grad} \mathrm{g}_m\left(\underline{\mathbf{s}}\right) = 0$$

Transmit Zero Forcing (ZF) (2)

with Wirtinger calculus

$$\underline{\mathbf{s}}^{*} - \sum_{m=1}^{M} \underline{\mathbf{H}}_{m}^{\mathrm{T}} \underline{\lambda}_{m} = 0$$



using $\underline{\boldsymbol{\lambda}} = \left(\underline{\lambda}_{1} \ldots \underline{\lambda}_{M}\right)^{\mathrm{T}}$

$$\underline{\mathbf{s}} - \underline{\mathbf{H}}^{*\mathrm{T}} \underline{\boldsymbol{\lambda}}^{*} = 0 \Rightarrow \underline{\mathbf{s}} = \underline{\mathbf{H}}^{*\mathrm{T}} \underline{\boldsymbol{\lambda}}^{*}$$

substitute in $\mathrm{g}(\underline{\mathbf{s}})$

$$\underline{\mathbf{d}} - \underline{\mathbf{H}}\underline{\mathbf{s}} = \underline{\mathbf{d}} - \underline{\mathbf{H}}\underline{\mathbf{H}}^{*\mathrm{T}} \underline{\boldsymbol{\lambda}}^{*} = 0 \Rightarrow \underline{\boldsymbol{\lambda}}^{*} = \left(\underline{\mathbf{H}}\underline{\mathbf{H}}^{*\mathrm{T}}\right)^{-1} \underline{\mathbf{d}}$$

$$\Rightarrow \underline{\mathbf{s}} = \underline{\mathbf{H}}^{*\mathrm{T}} \left(\underline{\mathbf{H}}\underline{\mathbf{H}}^{*\mathrm{T}}\right)^{-1} \underline{\mathbf{d}} \quad \Rightarrow \quad \boxed{\underline{\mathbf{M}}_{ZF} = \underline{\mathbf{H}}^{*\mathrm{T}} \left(\underline{\mathbf{H}}\underline{\mathbf{H}}^{*\mathrm{T}}\right)^{-1}}$$ right pseudoinverse

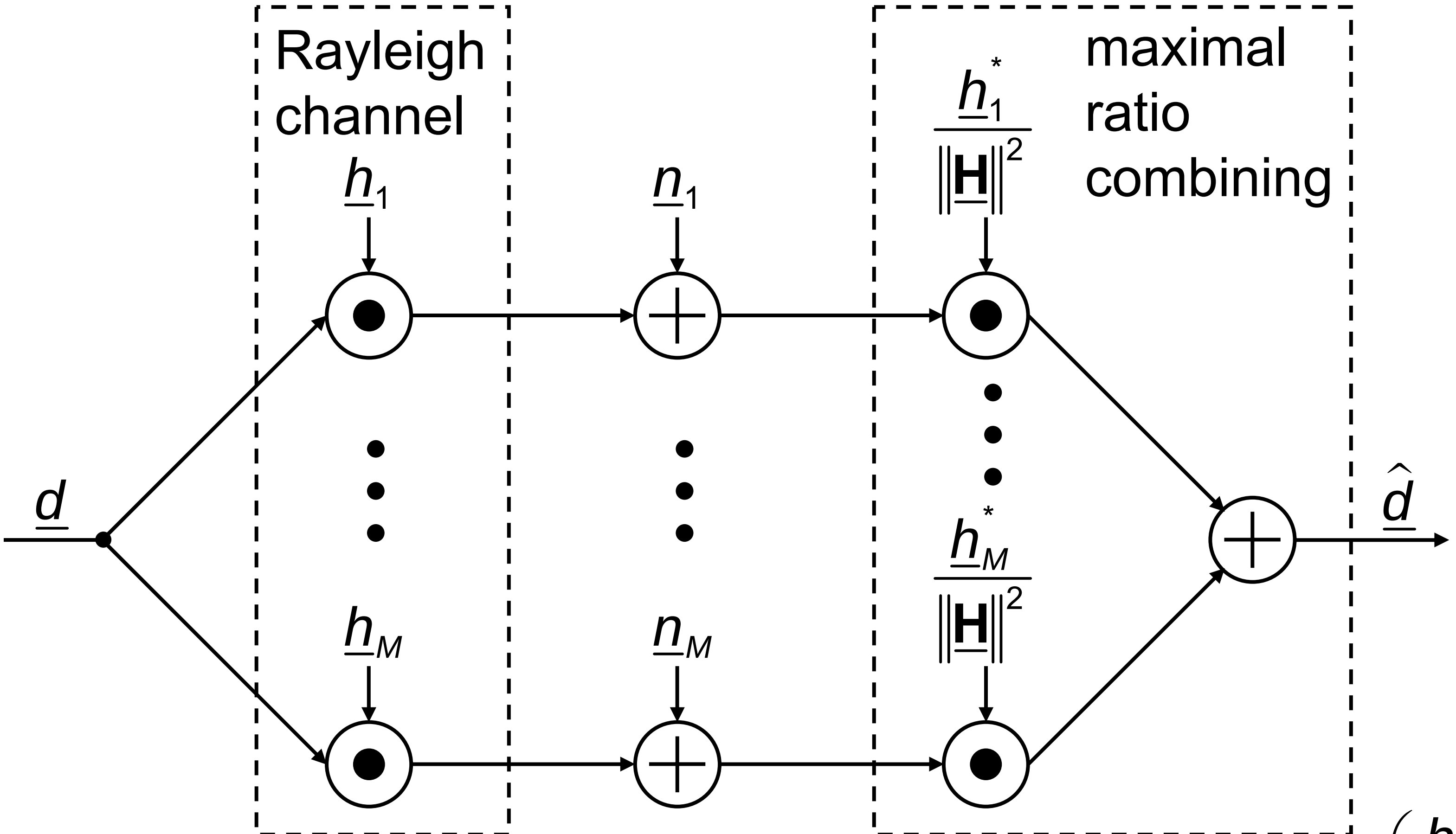
Diversity

transmission paths are unreliable

$\Rightarrow$ transmit information in parallel on several (independent) transmission paths

examples

- time diversity
- frequency diversity
- antenna diversity



σ^2: variance of the noise samples $\underline{n}_m$

σ_h^2: variance of the channel coefficients $\underline{h}_m$

$$\underline{\mathbf{H}} = \begin{pmatrix} \underline{h}_1 \\ \vdots \\ \underline{h}_M \end{pmatrix}$$

- channel energy $E_\mathrm{h} = \|\underline{\mathbf{H}}\|^2$ is chi square distributed with $2M$ degrees of freedom

$$\mathrm{p}(E_\mathrm{h}) = \frac{1}{\sigma_\mathrm{h}^{2M}(M-1)!} E_\mathrm{h}^{M-1} \cdot \mathrm{e}^{-\frac{E_\mathrm{h}}{\sigma_\mathrm{h}^2}}, \quad \overline{E}_\mathrm{h} = M\sigma_\mathrm{h}^2$$

- SNR of the estimated data symbol, QPSK modulation

$$\gamma = \frac{E_\mathrm{h} \cdot \mathrm{E}\left\{|\underline{d}|^2\right\}}{\sigma^2} = \frac{2E_\mathrm{h}}{\sigma^2}, \quad \overline{\gamma} = \frac{2\overline{E}_\mathrm{h}}{\sigma^2} = \frac{2M\sigma_\mathrm{h}^2}{\sigma^2}$$

$$\mathrm{p}(\gamma) = \frac{\sigma^{2M}\gamma^{M-1}}{\sigma_\mathrm{h}^{2M}2^M(M-1)!} \cdot \mathrm{e}^{-\frac{\gamma\sigma^2}{2\sigma_\mathrm{h}^2}} = \frac{\gamma^{M-1}M^M}{\overline{\gamma}^M(M-1)!} \cdot \mathrm{e}^{-\frac{M\gamma}{\overline{\gamma}}}$$

- bit error probability

$$P_\mathrm{b} = \frac{1}{2}\mathrm{erfc}\left(\sqrt{\frac{\gamma}{2}}\right), \quad \overline{P}_\mathrm{b} = \int_0^\infty P_\mathrm{b} \cdot \mathrm{p}(\gamma)\,\mathrm{d}\gamma$$

Average Bit Error Probability

- average bit error probability

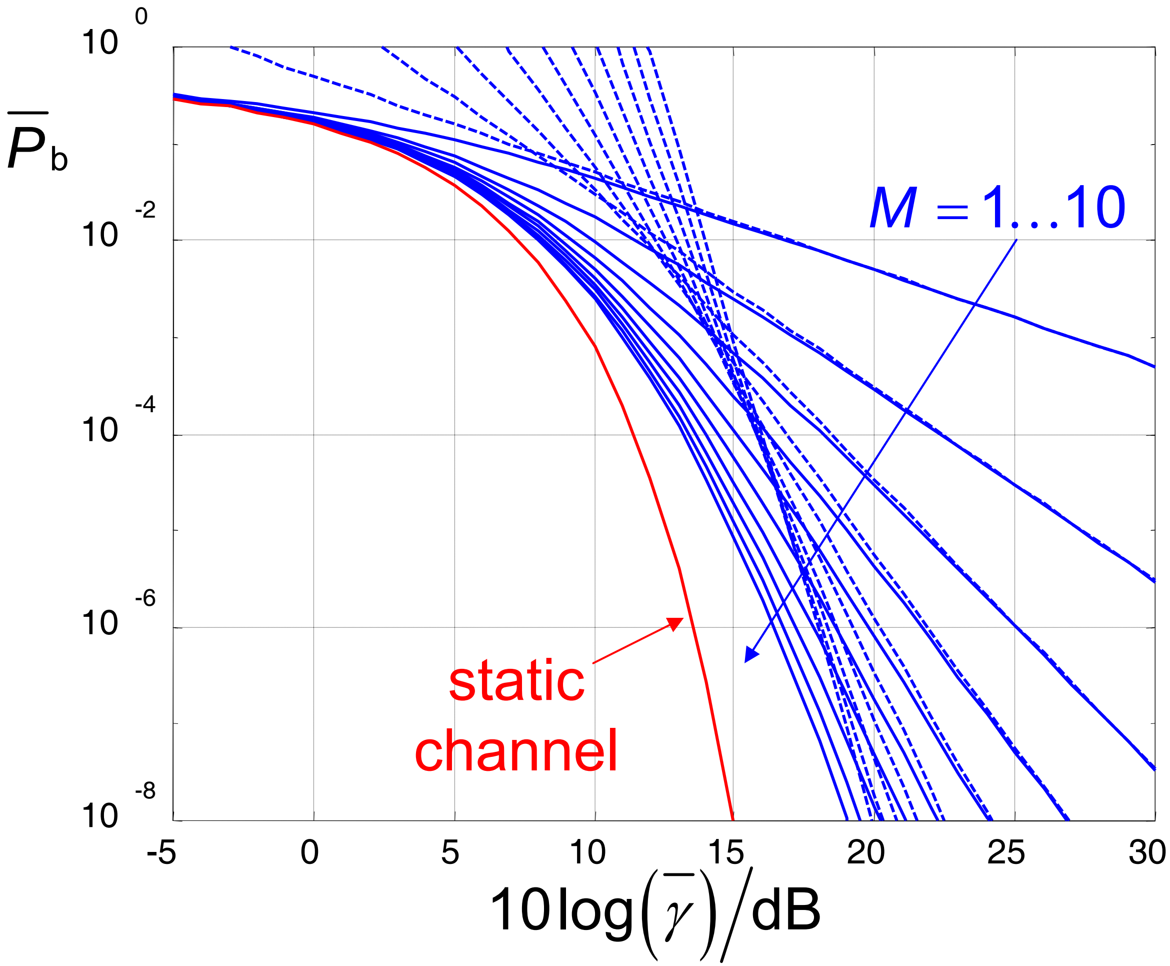
$$\overline{P}_b = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\overline{\gamma}}{\overline{\gamma}+2M}} \cdot \left(1 + \sum_{m=1}^{M-1} \frac{1\cdot 3 \cdot \ldots \cdot (2m-1)}{m!\,2^m \left(1+\frac{\overline{\gamma}}{2M}\right)^m}\right)$$

$$= \left(\frac{1}{2} - \frac{1}{2}\sqrt{\frac{\overline{\gamma}}{\overline{\gamma}+2M}}\right)^M \cdot \sum_{m=0}^{M-1} \binom{M-1+m}{m} \cdot \left(\frac{1}{2} + \frac{1}{2}\sqrt{\frac{\overline{\gamma}}{\overline{\gamma}+2M}}\right)^m$$

- for large average SNR $\overline{\gamma}$

$$\overline{P}_b = \binom{2M-1}{M} \cdot \left(\frac{M}{2\overline{\gamma}}\right)^M \sim \frac{1}{\overline{\gamma}^M}$$

Bit Error Performance



Diversity Degree

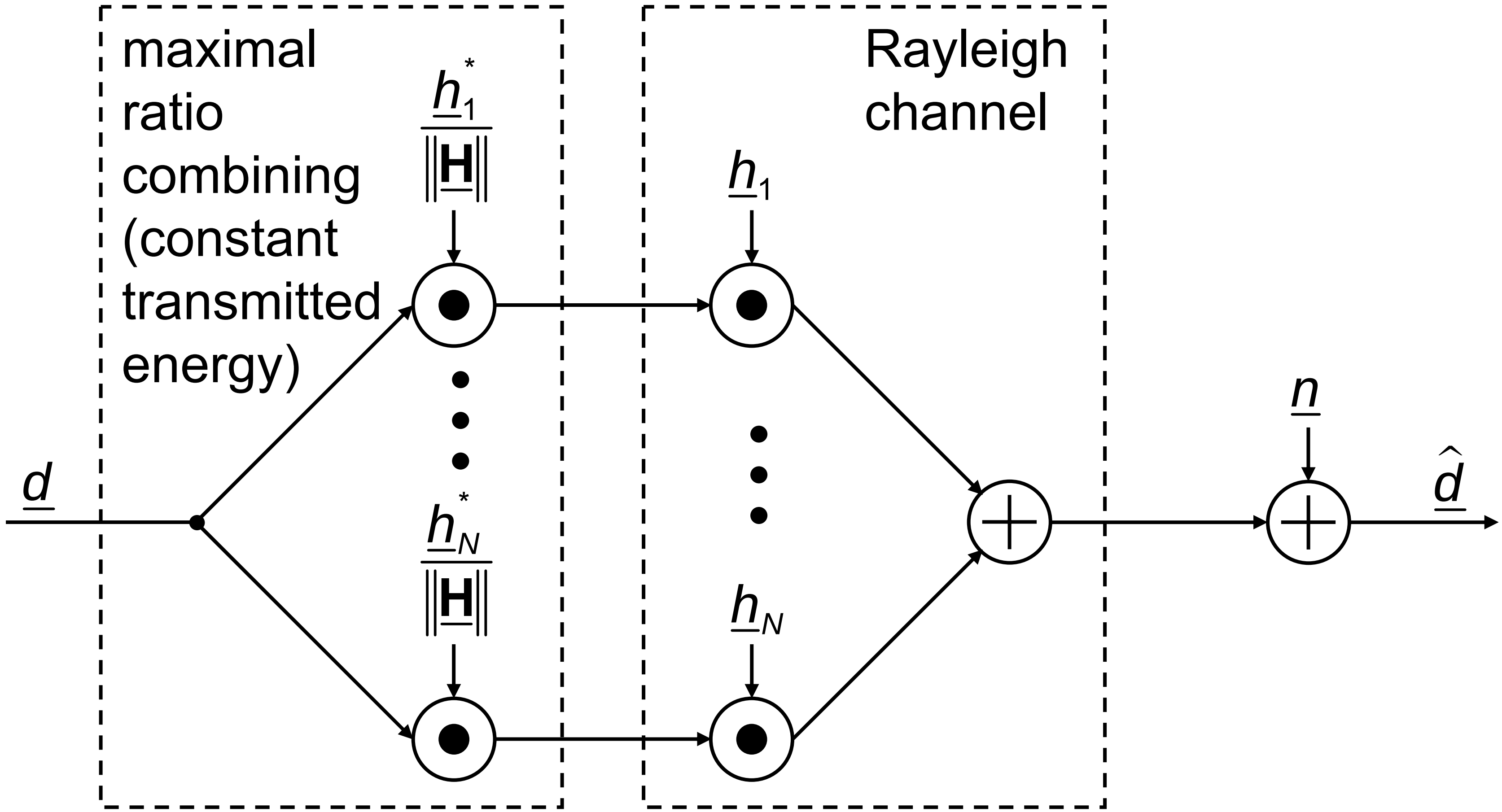
consider the negative asymptotic slope of the bit error probability curve

$\Rightarrow$ diversity degree

$$D = -\lim_{\gamma \to \infty} \frac{\log\left(\overline{P}_{\mathrm{b}}\right)}{\log\left(\overline{\gamma}\right)}$$

here: $D = M$

Transmit Diversity



σ^2: variance of the noise $\underline{n}$

σ_h^2: variance of the channel coefficients $\underline{h}_n$

$$\underline{\mathbf{H}} = \begin{pmatrix} \underline{h}_1 & \dots & \underline{h}_N \end{pmatrix}$$

Analysis

- channel energy $E_\mathrm{h} = \|\underline{\mathbf{H}}\|^2$ is chi square distributed with 2*N* degrees of freedom

$$\mathrm{p}(E_\mathrm{h}) = \frac{1}{\sigma_\mathrm{h}^{2N}(N-1)!} E_\mathrm{h}^{N-1} \cdot \mathrm{e}^{-\frac{E_\mathrm{h}}{\sigma_\mathrm{h}^2}}, \quad \overline{E}_\mathrm{h} = N\sigma_\mathrm{h}^2$$

- SNR of the estimated data symbol, QPSK modulation

$$\gamma = \frac{2E_\mathrm{h}}{\sigma^2}, \quad \overline{\gamma} = \frac{2N\sigma_\mathrm{h}^2}{\sigma^2}, \quad \mathrm{p}(\gamma) = \frac{\sigma^{2N}\gamma^{N-1}}{\sigma_\mathrm{h}^{2N}2^N(N-1)!} \cdot \mathrm{e}^{-\frac{\gamma\sigma^2}{2\sigma_\mathrm{h}^2}}$$

- bit error probability

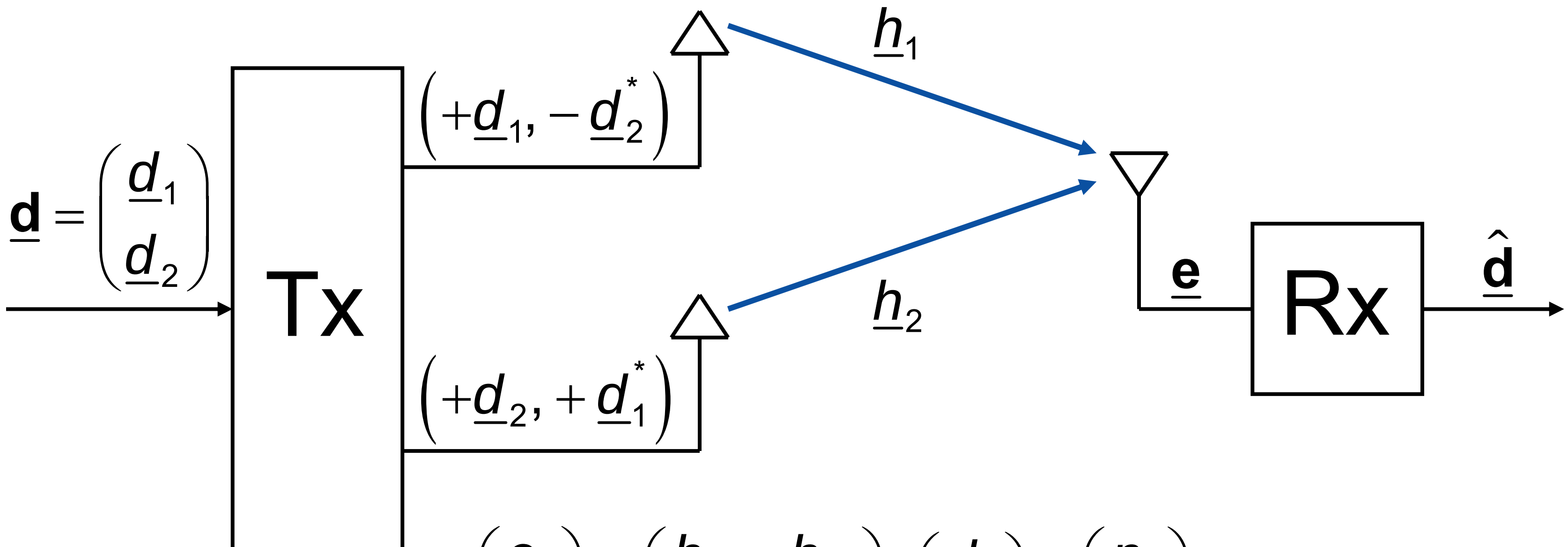
$$P_\mathrm{b} = \frac{1}{2}\mathrm{erfc}\left(\sqrt{\frac{\gamma}{2}}\right), \quad \overline{P}_\mathrm{b} = \int_0^\infty P_\mathrm{b} \cdot \mathrm{p}(\gamma)\,\mathrm{d}\gamma$$

$\Rightarrow$ same performance as receive diversity, diversity degree *D* = *N*

Alamouti Scheme

Transmit Diversity can be exploited without transmitter side channel state information!

Alamouti (1998): *A Simple Transmit Diversity Technique for Wireless Communications*



$$\underbrace{\begin{pmatrix} \underline{e}_1 \\ \underline{e}_2^* \end{pmatrix}}_{\underline{\mathbf{e}}} = \underbrace{\begin{pmatrix} \underline{h}_1 & \underline{h}_2 \\ \underline{h}_2^* & -\underline{h}_1^* \end{pmatrix}}_{\underline{\mathbf{H}}} \cdot \underbrace{\begin{pmatrix} \underline{d}_1 \\ \underline{d}_2 \end{pmatrix}}_{\underline{\mathbf{d}}} + \underbrace{\begin{pmatrix} \underline{n}_1 \\ \underline{n}_2^* \end{pmatrix}}_{\underline{\mathbf{n}}}$$

Maximum Likelihood Receiver

- the columns of the system matrix $\underline{\mathbf{H}}$ are orthogonal
  $\Rightarrow$ optimum receiver consists in a matched filter followed by a quantizer

$$\hat{\underline{\mathbf{d}}} = \left(\operatorname{diag}\left(\underline{\mathbf{H}}^{*T}\underline{\mathbf{H}}\right)\right)^{-1}\underline{\mathbf{H}}^{*T}\begin{pmatrix}\underline{e}_1\\ \underline{e}_2^*\end{pmatrix} = \frac{1}{|\underline{h}_1|^2+|\underline{h}_2|^2}\begin{pmatrix}\underline{h}_1^* & \underline{h}_2\\ \underline{h}_2^* & -\underline{h}_1\end{pmatrix}\cdot\begin{pmatrix}\underline{e}_1\\ \underline{e}_2^*\end{pmatrix}$$

$$= \begin{pmatrix}\underline{d}_1\\ \underline{d}_2\end{pmatrix} + \frac{1}{|\underline{h}_1|^2+|\underline{h}_2|^2}\begin{pmatrix}\underline{h}_1^*\underline{n}_1+\underline{h}_2\underline{n}_2^*\\ \underline{h}_2^*\underline{n}_1-\underline{h}_1\underline{n}_2^*\end{pmatrix}$$

- SNR of the estimates for QPSK modulation $\quad \mathrm{E}\left\{\left\|\underline{d}_n\right\|^2\right\} = 2$

$$\gamma = 2\frac{|\underline{h}_1|^2+|\underline{h}_2|^2}{\sigma^2}$$

Same SNR as for transmit diversity with transmitter side channel state information but twice the transmit power because there is no beam forming gain!

Rate

Similar to the concept of the rate used in coding theory one defines the rate of a spatio temporal code:

$$r = \frac{\text{number of data symbols}}{\text{number of channel uses}}$$

examples:

- spread spectrum system, spreading factor Q (repetition code)
  $$r = \frac{1}{Q} \leq 1$$
- Alamouti code
  $$r = \frac{2}{2} = 1$$
- spatial multiplexing
  $$r = \min(N, M) \geq 1$$

End